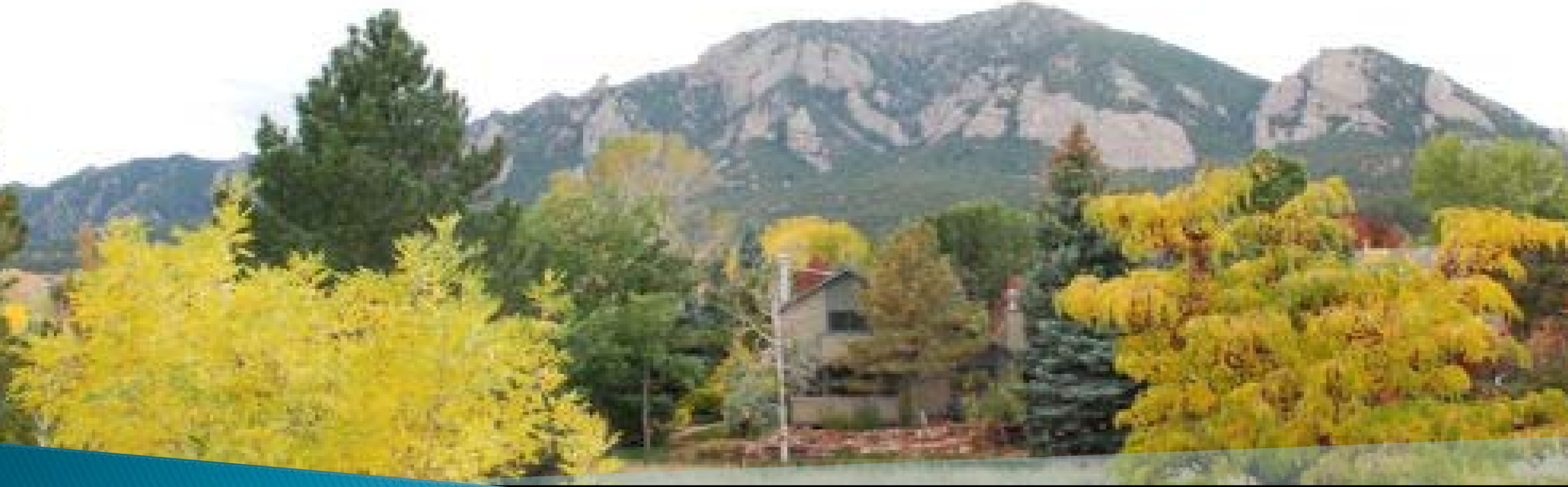
A Proposed $(k-\epsilon)^2$ Turbulence Theory for Multiphase Gas-Solid Flows

Reynolds Averaged Navier-Stokes (RANS) Theory for the Gas-Particle Two-Fluid Model



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In memoriam
Prof. Adel Sarofim
1934-2011

- Chemistry BA (1955), Oxford
- Chemical Engineering Practice SM (1957) and Chemical Engineering ScD (1962), MIT
- Prof. emeritus MIT, Dept. Chem. Eng.
- Presidential Prof., Univ. Utah
- Co-founder & Senior Technical Advisor, Reaction Engineering International (REI)

In memoriam
Prof. Adel Sarofim
1934-2011

"… a compassionate human being who inspired students and colleagues, and who contributed significantly across the full spectrum from fundamental science through real-world design concepts." U.S. DoE Homer H. Lowry Award (1996)

"… for advancing our understanding of the mechanisms and modeling of processes that control radiation in and pollution emission from combustors." National Academy of Engineering (2003)

Outline

- A little rambling
- Favre-like average
  - Definitions
  - Identities
- Averaged equations
  - Derivation
  - Closures
- Turbulent kinetic energy equations
  - Derivation
  - Closures

Model Hierarchy

Direct numerical simulation (DNS)

fluid: Navier–Stokes equation
particles: Newton's equations of motion
interaction: traction on the surface of the particles

Discrete element modeling (DEM)

fluid: Navier–Stokes equation +
particles: Newton's equations of motion
interaction: empirical correlation

Two–fluid model (e.g., MFIX)

fluids: Navier–Stokes equation +
particles: Navier–Stokes–like equation +
interaction: empirical correlation

Model Hierarchy

Direct numerical simulation (DNS)

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Two-fluid model (e.g., MFIX)

fluids: Navier-Stokes equation +
particles: Navier-Stokes-like equation +
interaction: empirical correlation

Reynolds-averaged Two-fluid model (RANS: RAT-F; FAT-F)

fluids: **time-steady** Navier-Stokes equation +
particles: **time-steady** Navier-Stokes-like equation +
interaction: empirical correlation

Previous work

Elghobashi, S.E., and T.W. Abou-Arab, "A two equation turbulence model for two- phase flows," *Phys. Fluids* **26**, 931–938, 1983.

Besnard, D.C., and F.H. Harlow, "Turbulence in multiphase flow," *Int. J. Multiphase Flow* **14**, 679–699, 1988.

Kashiwa, B.A., and W.B. VanderHeyden, "Toward a general theory for multiphase turbulence Part I: Development and gauging of the model equations," LA-13773-MS, Dec. 2000.

Krepper, E., D. Lucas and H.-M. Prasser, "On the modelling of bubbly flow in vertical pipes," *Nucl. Eng. Des.* **235**, 597–611, 2005.

Krepper, E., M. Beyer, T. Frank, D. Lucas and H.-M. Prasser, "CFD modelling of polydispersed bubbly two-phase flow around an obstacle," *Nucl. Eng. Des.* **239**, 2372–2381, 2009.

Lahey Jr., R.T., and D.A. Drew, "The analysis of two-phase flow and heat transfer using a multidimensional, four-field, two-fluid model," *Nucl. Eng. Des.* **204**, 29–44, 2001.

Lahey Jr., R.T., "The simulation of multidimensional multiphase flows," *Nucl. Eng. Des.* **235**, 1043–1060, 2005.

MFIX Governing Equations
(Continuity equations)

Continuity equation for gas phase, *g*:

$$\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right) + \frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right) = 0$$

Continuity equation for solids phase, *s*:

$$\frac{\partial}{\partial t}\left(\varepsilon_s \rho_s\right) + \frac{\partial}{\partial x_i}\left(\varepsilon_s \rho_s U_{si}\right) = 0$$

MFIX Governing Equations
(Momentum equations)

Momentum equations for gas phase, *g*:

$$\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g U_{gi}\right)+\frac{\partial}{\partial x_j}\left(\varepsilon_g \rho_g U_{gj} U_{gi}\right)$$

$$= -\varepsilon_g \frac{\partial P_g}{\partial x_i} + \frac{\partial \tau_{gij}}{\partial x_j} - I_{gsi} + \varepsilon_g \rho_g g_i$$

Momentum equations for solids phase, *s*:

$$\frac{\partial}{\partial t}\left(\varepsilon_s \rho_s U_{si}\right)+\frac{\partial}{\partial x_j}\left(\varepsilon_s \rho_s U_{sj} U_{si}\right)$$

$$= -\varepsilon_s \frac{\partial P_g}{\partial x_i} + \frac{\partial \tau_{sij}}{\partial x_j} + I_{gsi} + \varepsilon_S \rho_S g_i$$

Favre & Favre-like

Favre average: compressible gas

$$\tilde{U}_{gi} \equiv \overline{\rho_g U_{gi}} \Big/ \rho_g$$

Favre, A., "Équations des gaz turbulents compressibles I.- Formes générales," Journal de Mécanique **4**, 361-390, 1965.

Favre, A., "Formulation of the statistical equations of turbulent flows with variable density," in *Studies in Turbulence*, T.B. Gatski, S. Sarkar and C.O. Speziale eds., Springer-Verlag, 324-341, 1992.

Favre-like average: two-fluid

$$\tilde{U}_{si} \equiv \overline{\rho_{s,\mathrm{bulk}} U_{si}} \Big/ \overline{\rho}_{s,\mathrm{bulk}}$$

$$= \overline{\varepsilon_s \rho_s U_{si}} \Big/ \left(\overline{\varepsilon_s \rho_s} \right)$$

$$= \overline{\varepsilon_s U_{si}} \Big/ \overline{\varepsilon}_s$$

$$\tilde{U}_{gi} \equiv \overline{\varepsilon_g U_{gi}} \Big/ \overline{\varepsilon}_g$$

Notation

Dependent variables

$$\varepsilon_s, \varepsilon_g, U_{si}, U_{gi}, P_g$$

Favre-like average

$$\tilde{U}_{si} \equiv \overline{\varepsilon_s U_{si}} \Big/ \bar{\varepsilon}_s \,, \tilde{U}_{gi} \equiv \overline{\varepsilon_g U_{gi}} \Big/ \bar{\varepsilon}_g$$

Reynolds decomposition

$$\varepsilon_s = \bar{\varepsilon}_s + \varepsilon'_s$$

$$\varepsilon_g = \bar{\varepsilon}_g + \varepsilon'_g$$

$$U_{si} = \overline{U}_{si} + U'_{si}$$

$$U_{gi} = \overline{U}_{gi} + U'_{gi}$$

$$P_g = \overline{P}_g + P'_g$$

Favre decomposition

$$\cdots$$

$$\cdots$$

$$U_{si} = \tilde{U}_{si} + U''_{si}$$

$$U_{gi} = \tilde{U}_{gi} + U''_{gi}$$

$$\cdots$$

Identities
… following from the definitions

$$\overline{\varepsilon'_s}(\mathbf{x}) = \overline{\varepsilon'_g} = \overline{U'_{si}} = \overline{U'_{gi}} = \overline{P'_g} = 0 \qquad \varepsilon'_g(\mathbf{x},t) = -\varepsilon'_s(\mathbf{x},t)$$

$$\overline{\varepsilon_g U''_{gi}} = 0 \left(\overline{U''_{gi}} \neq 0\right) \qquad \overline{\varepsilon_s U''_{si}} = 0 \left(\overline{U''_{si}} \neq 0\right)$$

$$\overline{U}_{gi} - \tilde{U}_{gi} = \overline{U''_{gi}} = -\tilde{U}'_{gi} \qquad \overline{U}_{si} - \tilde{U}_{si} = \overline{U''_{si}} = -\tilde{U}'_{si}$$

$$\overline{U''_{gi}} = -\tilde{U}'_{gi} = -\overline{\varepsilon'_g U''_{gi}} \Big/ \overline{\varepsilon}_g \qquad \overline{U''_{si}} = -\tilde{U}'_{si} = -\overline{\varepsilon'_s U''_{si}} \Big/ \overline{\varepsilon}_s$$

Reynolds/Favre Averaged Gas Phase Continuity Equation

Instantaneous, local gas phase continuity equation

$$\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right) + \frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right) = 0$$

Take the time average

$$\overline{\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right)} + \overline{\frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right)} = 0$$

Reynolds/Favre Averaged Gas Phase Continuity Equation

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Take the time average

$$\overline{\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right)} + \overline{\frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right)} = 0$$

Time averaging commutes with both temporal and spatial differentiation

$$\frac{\partial}{\partial t}\left(\rho_g \overline{\varepsilon}_g\right) + \frac{\partial}{\partial x_i}\left(\rho_g \overline{\varepsilon_g U_{gi}}\right) = 0$$

Reynolds/Favre Averaged Gas Phase Continuity Equation

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$$\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right) + \frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right) = 0$$

Take the time average

$$\overline{\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right)} + \overline{\frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right)} = 0$$

Time averaging commutes with both temporal and spatial differentiation

$$\frac{\partial}{\partial t}\left(\rho_g \bar{\varepsilon}_g\right) + \frac{\partial}{\partial x_i}\left(\rho_g \overline{\varepsilon_g U_{gi}}\right) = 0$$

Use $\overline{\varepsilon_g U_{gi}} = \bar{\varepsilon}_g \tilde{U}_{gi}$

$$\frac{\partial}{\partial t}\left(\rho_g \bar{\varepsilon}_g\right) + \boxed{\frac{\partial}{\partial x_i}\left(\rho_g \bar{\varepsilon}_g \tilde{U}_{gi}\right)} = 0$$

Reynolds Averaged Gas Phase Continuity Equation

$$\frac{\partial}{\partial t}\left(\rho_g \bar{\varepsilon}_g\right)+\frac{\partial}{\partial x_i}\left(\rho_g \overline{\varepsilon_g U_{gi}}\right)=0$$

Expand $\overline{\varepsilon_g U_{gi}} = \overline{\left(\bar{\varepsilon}_g+\varepsilon'_g\right)\left(\overline{U}_{gi}+U'_{gi}\right)} = \bar{\varepsilon}_g \overline{U}_{gi} + \bar{\varepsilon}_g \overline{U'_{gi}} + \overline{\varepsilon'_g}\,\overline{U}_{gi} + \overline{\varepsilon'_g U'_{gi}}$

$$= \bar{\varepsilon}_g \overline{U}_{gi} + \overline{\varepsilon'_g U'_{gi}}$$

So

$$\frac{\partial}{\partial t}\left(\rho_g \bar{\varepsilon}_g\right)+\frac{\partial}{\partial x_i}\left(\rho_g\left(\bar{\varepsilon}_g \overline{U}_{gi}+\overline{\varepsilon'_g U'_{gi}}\right)\right)=0$$

Compared with R/F averaged

$$\frac{\partial}{\partial t}\left(\rho_g \bar{\varepsilon}_g\right)+\frac{\partial}{\partial x_i}\left(\rho_g \bar{\varepsilon}_g \tilde{U}_{gi}\right)=0$$

But,

$$\tilde{U}_{gi}=\overline{U}_{gi}+\overline{\varepsilon'_g U'_{gi}}\Big/\bar{\varepsilon}_g$$

Continuity equations

Gas phase

$$\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g\right) + \frac{\partial}{\partial x_i}\left(\varepsilon_g \rho_g U_{gi}\right) = 0$$

$$\cdots \Rightarrow \quad \underline{\underline{\frac{\partial}{\partial t}\left(\rho_g\, \bar{\varepsilon}_g\right) + \frac{\partial}{\partial x_i}\left(\rho_g\, \bar{\varepsilon}_g\, \tilde{U}_{gi}\right) = 0}}$$

Solids phase

$$\frac{\partial}{\partial t}\left(\varepsilon_s \rho_s\right) + \frac{\partial}{\partial x_i}\left(\varepsilon_s \rho_s U_{si}\right) = 0$$

$$\cdots \Rightarrow \quad \underline{\underline{\frac{\partial}{\partial t}\left(\rho_s\, \bar{\varepsilon}_s\right) + \frac{\partial}{\partial x_i}\left(\rho_s\, \bar{\varepsilon}_s\, \tilde{U}_{si}\right) = 0}}$$

Gas Phase Momentum Equations

Instantaneous, local gas phase momentum equations

$$\frac{\partial}{\partial t}\left(\varepsilon_g \rho_g U_{gi}\right) + \frac{\partial}{\partial x_j}\left(\varepsilon_g \rho_g U_{gj} U_{gi}\right)$$

$$= -\varepsilon_g \frac{\partial P_g}{\partial x_i} + \frac{\partial \tau_{gij}}{\partial x_j} - I_{gsi} + \varepsilon_g \rho_g g_i$$

Take the time average (commutes with differentiation)

$$\frac{\partial}{\partial t}\left(\rho_g \overline{\varepsilon_g U_{gi}}\right) + \frac{\partial}{\partial x_j}\left(\rho_g \overline{\varepsilon_g U_{gj} U_{gi}}\right)$$

done Much work

$$= -\overline{\varepsilon_g \frac{\partial P_g}{\partial x_i}} + \frac{\partial \bar{\tau}_{gij}}{\partial x_j} - \bar{I}_{gsi} + \bar{\varepsilon}_g \rho_g g_i$$

new easy

Gas phase momentum equation
convective triple product – Favre average

$$\varepsilon_g U_{gj} U_{gi} = \varepsilon_g \left(\tilde{U}_{gj} + U''_{gj}\right)\left(\tilde{U}_{gi} + U''_{gi}\right)$$

$$= \varepsilon_g \tilde{U}_{gj} \tilde{U}_{gi} + \varepsilon_g U''_{gi} \tilde{U}_{gj} + \varepsilon_g U''_{gj} \tilde{U}_{gi} + \varepsilon_g U''_{gj} U''_{gi}$$

Gas phase momentum equation

convective triple product – Favre average

$$\varepsilon_g U_{gj} U_{gi} = \varepsilon_g \left(\tilde{U}_{gj} + U''_{gj}\right)\left(\tilde{U}_{gi} + U''_{gi}\right)$$

$$= \varepsilon_g \tilde{U}_{gj} \tilde{U}_{gi} + \varepsilon_g U''_{gi} \tilde{U}_{gj} + \varepsilon_g U''_{gj} \tilde{U}_{gi} + \varepsilon_g U''_{gj} U''_{gi}$$

time average

$$\overline{\varepsilon_g U_{gj} U_{gi}}$$

$$= \bar{\varepsilon}_g \tilde{U}_{gj} \tilde{U}_{gi} + \cancel{\overline{\varepsilon_g U''_{gi}}} \tilde{U}_{gj} + \cancel{\overline{\varepsilon_g U''_{gj}}} \tilde{U}_{gi} + \overline{\varepsilon_g U''_{gj} U''_{gi}}$$

Gas phase momentum equation

convective triple product – Favre average

$$\overline{\varepsilon_g U_{gj} U_{gi}} = \bar{\varepsilon}_g \tilde{U}_{gj} \tilde{U}_{gi} + \overline{\varepsilon_g U''_{gj} U''_{gi}}$$

$$= \bar{\varepsilon}_g \overline{\overline{U}}_{gj} \overline{\overline{U}}_{gi} + \bar{\varepsilon}_g \overline{\overline{U''_{gj} U''_{gi}}}$$

$$= \bar{\varepsilon}_g \tilde{U}_{gj} \tilde{U}_{gi} - \bar{\varepsilon}_g \tau_{\tilde{T}gji}$$

gas phase specific Reynolds - ?? stresses

$$\tau_{\tilde{T}gji} \equiv -\overline{\overline{U''_{gj} U''_{gi}}}$$

$$= -\overline{\varepsilon_g U''_{gj} U''_{gi}} \Big/ \bar{\varepsilon}_g$$

$$= -\overline{U''_{gj} U''_{gi}} - \overline{\varepsilon'_g U''_{gj} U''_{gi}} \Big/ \bar{\varepsilon}_g$$

Gas phase momentum equation

convective triple product – Reynolds average

$$\overline{\varepsilon_g U_{gj} U_{gi}} = \overline{\left(\bar{\varepsilon}_g + \varepsilon'_g\right)\left(\overline{U}_{gj} + U'_{gj}\right)\left(\overline{U}_{gi} + U'_{gi}\right)}$$

$$= \bar{\varepsilon}_g \overline{U}_{gj}\,\overline{U}_{gi} + \bar{\varepsilon}_g \overline{U'_{gi}}\,\overline{U}_{gj} + \bar{\varepsilon}_g \overline{U'_{gj}}\,\overline{U}_{gi} + \bar{\varepsilon}_g \overline{U'_{gj} U'_{gi}}$$
$$+ \overline{\varepsilon'_g}\,\overline{U}_{gj}\,\overline{U}_{gi} + \overline{\varepsilon'_g U'_{gi}}\,\overline{U}_{gj} + \overline{\varepsilon'_g U'_{gj}}\,\overline{U}_{gi} + \overline{\varepsilon'_g U'_{gj} U'_{gi}}$$

$$= \bar{\varepsilon}_g \overline{U}_{gj}\,\overline{U}_{gi} + \bar{\varepsilon}_g \overline{U'_{gj} U'_{gi}} + \overline{\varepsilon'_g U'_{gi}}\,\overline{U}_{gj} + \overline{\varepsilon'_g U'_{gj}}\,\overline{U}_{gi} + \overline{\varepsilon'_g U'_{gj} U'_{gi}}$$

$$cf. \quad = \bar{\varepsilon}_g \tilde{U}_{gj} \tilde{U}_{gi} - \bar{\varepsilon}_g \tau_{\tilde{T}gji}$$

$$\tau_{\tilde{T}gji} \equiv -\overline{\varepsilon_g U''_{gj} U''_{gi}} \Big/ \bar{\varepsilon}_g = -\overline{U''_{gj} U''_{gi}} - \overline{\varepsilon'_g U''_{gj} U''_{gi}} \Big/ \bar{\varepsilon}_g$$

Reynolds average *cf.* Reynolds/Favre average
Convective Term

Reynolds averaging

$$\overline{U'_{gj}\,U'_{gi}}$$

$$\overline{\varepsilon'_g\,U'_{gi}}$$

$$\overline{\varepsilon'_g\,U'_{gj}\,U'_{gi}}$$

Favre averaging

$$\tau_{\tilde{T}gji} \equiv -\overline{\overline{U''_{gj}\,U''_{gi}}} = -\overline{\varepsilon_g\,U''_{gj}\,U''_{gi}}\Big/\overline{\varepsilon}_g$$

$$= -\overline{U''_{gj}\,U''_{gi}} - \overline{\varepsilon'_g\,U''_{gj}\,U''_{gi}}\Big/\overline{\varepsilon}_g$$

Gas phase momentum equation
convective term

$$\overline{\frac{\partial}{\partial x_j}\left(\varepsilon_g \rho_g U_{gj} U_{gi}\right)}$$

$$= \frac{\partial}{\partial x_j}\left(\varepsilon_g \overline{\rho_g U_{gj} U_{gi}}\right)$$

$$= \frac{\partial}{\partial x_j}\left(\bar{\varepsilon}_g \tilde{U}_{gj} \tilde{U}_{gi} - \bar{\varepsilon}_g \tilde{\tau}_{Tgji}\right)$$

gas phase specific Reynolds stresses

$$\tilde{\tau}_{Tgji} \equiv -\overline{\overline{U''_{gj} U''_{gi}}} = -\overline{\varepsilon_g U''_{gj} U''_{gi}} \Big/ \bar{\varepsilon}_g$$

$$= -\overline{U''_{gj} U''_{gi}} - \overline{\varepsilon'_g U''_{gj} U''_{gi}} \Big/ \bar{\varepsilon}_g$$

Gas phase momentum equation
interaction term – an empirical 2-fluid closure

The general form is written as

$$I_{gsi} = \varepsilon_s \hat{\beta}_{gs} \left(U_{gi} - U_{si} \right)$$

The coefficient is represented as

$$\hat{\beta}_{gs} = \overline{\hat{\beta}}_{gs} \left(\mathbf{x} \right) + \hat{\beta}'_{gs} \left(\mathbf{x}, t \right)$$

so that the time average can be written as

$$\begin{aligned} \bar{I}_{gsi} &= \overline{\hat{\beta}}_{gs} \left(\overline{\varepsilon_s U_{gi}} - \overline{\varepsilon_s U_{si}} \right) + \overline{\hat{\beta}'_{gs} \left(\varepsilon_s U_{gi} - \varepsilon_s U_{si} \right)} \\ &= \bar{I}_{gsi-1} + \bar{I}_{gsi-2} \end{aligned}$$

Gas phase momentum equation
interaction term– cont

$$\bar{I}_{gsi-1} = \bar{\hat{\beta}}_{gs}\left(\overline{\varepsilon_s U_{gi}} - \overline{\varepsilon_s U_{si}}\right)$$

Clearly $\overline{\varepsilon_s U_{si}} = \bar{\varepsilon}_s \tilde{U}_{si}$ but $\overline{\varepsilon_s U_{gi}}$ is problematic:

$$\overline{\varepsilon_s U_{gi}} = \overline{\left(1-\varepsilon_g\right) U_{gi}} = \overline{U_{gi}} - \overline{\varepsilon_g U_{gi}}$$

$$= \overline{U_{gi}} - \bar{\varepsilon}_g \tilde{U}_{gi} = \overline{U_{gi}} - \left(1-\bar{\varepsilon}_s\right)\tilde{U}_{gi} = \overline{U_{gi}} - \tilde{U}_{gi} + \bar{\varepsilon}_s \tilde{U}_{gi}$$

Using the identity $\overline{U_{gi}} - \tilde{U}_{gi} = \overline{U''_{gi}}$,

$$\overline{\varepsilon_s U_{gi}} = \overline{U''_{gi}} + \bar{\varepsilon}_s \tilde{U}_{gi}$$

so that $\underline{\underline{\bar{I}_{gsi-1} = \bar{\varepsilon}_s \bar{\hat{\beta}}_{gs}\left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \bar{\hat{\beta}}_{gs}\overline{U''_{gi}}}}$

Gas phase momentum equation
interaction term– cont again

Examining the second term more thoroughly,

$$I_{gsi-2} = \hat{\beta}'_{gs}\varepsilon_s\left(U_{gi} - U_{si}\right) = \hat{\beta}'_{gs}\left(\bar{\varepsilon}_s + \varepsilon'_s\right)\left(\tilde{U}_{gi} + U''_{gi} - \tilde{U}_{si} - U''_{si}\right)$$

$$= \hat{\beta}'_{gs}\bar{\varepsilon}_s\left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \hat{\beta}'_{gs}\bar{\varepsilon}_s\left(U''_{gi} - U''_{si}\right)$$

$$+ \hat{\beta}'_{gs}\,\varepsilon'_s\left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \hat{\beta}'_{gs}\,\varepsilon'_s\left(U''_{gi} - U''_{si}\right)$$

Using $\varepsilon'_s = -\varepsilon'_g$

$$\overline{\hat{\beta}'_{gs}\varepsilon_s\left(U_{gi} - U_{si}\right)} = \bar{\varepsilon}_s\,\overline{\hat{\beta}'_{gs}\left(U''_{gi} - U''_{si}\right)} + \overline{\hat{\beta}'_{gs}\,\varepsilon'_s}\left(\tilde{U}_{gi} - \tilde{U}_{si}\right)$$

$$+ \overline{\hat{\beta}'_{gs}\left(\varepsilon'_g U''_{gi} + \varepsilon'_s U''_{si}\right)}$$

Gas phase momentum equation
interaction term– finally

$$\bar{I}_{gsi} = \bar{\hat{\beta}}_{gs}\,\bar{\varepsilon}_s\left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \bar{\hat{\beta}}_{gs}\overline{U''_{gi}}$$
$$+\ \overline{\hat{\beta}'_{gs}\,\varepsilon'_s\left(\tilde{U}_{gi} - \tilde{U}_{si}\right)} + \bar{\varepsilon}_s\overline{\hat{\beta}'_{gs}\left(U''_{gi} - U''_{si}\right)}$$
$$-\ \overline{\hat{\beta}'_{gs}\left(\varepsilon'_g U''_{gi} + \varepsilon'_s U''_{si}\right)}.$$

Gas phase momentum equation
pressure term

$$\varepsilon_g \frac{\partial P_g}{\partial x_i} = \left(\overline{\varepsilon}_g + \varepsilon'_g\right)\frac{\partial\left(\overline{P}_g + P'_g\right)}{\partial x_i} = \overline{\varepsilon}_g \frac{\partial \overline{P}_g}{\partial x_i} + \overline{\varepsilon}_g \frac{\partial P'_g}{\partial x_i} + \varepsilon'_g \frac{\partial \overline{P}_g}{\partial x_i} + \varepsilon'_g \frac{\partial P'_g}{\partial x_i}$$

$$\overline{\varepsilon_g \frac{\partial P_g}{\partial x_i}} = \overline{\varepsilon}_g \frac{\partial \overline{P}_g}{\partial x_i} + \overline{\varepsilon}_g \frac{\partial \overline{P'_g}}{\partial x_i} + \overline{\varepsilon'_g} \frac{\partial \overline{P}_g}{\partial x_i} + \overline{\varepsilon'_g \frac{\partial P'_g}{\partial x_i}}$$

$$\underline{\underline{\overline{\varepsilon_g \frac{\partial P_g}{\partial x_i}} = \overline{\varepsilon}_g \frac{\partial \overline{P}_g}{\partial x_i} + \overline{\varepsilon'_g \frac{\partial P'_g}{\partial x_i}}}}$$

voidage - pressure correlation

$$\overline{\varepsilon'_g \frac{\partial P'_g}{\partial x_i}}$$

Gas phase momentum equation

stress term

$$\bar{\tau}_{gij} = 2\mu_g \bar{S}_{gij} = 2\mu_g \left[\frac{1}{2}\left(\frac{\partial\left(\tilde{U}_{gi} + \overline{U''_{gi}}\right)}{\partial x_j} + \frac{\partial\left(\tilde{U}_{gj} + \overline{U''_{gj}}\right)}{\partial x_i} \right) - \frac{1}{3}\frac{\partial\left(\tilde{U}_{gi} + \overline{U''_{gi}}\right)}{\partial x_i}\delta_{ij} \right]$$

$$\bar{\tau}_{gij} = 2\mu_g \left[\frac{1}{2}\left(\frac{\partial \tilde{U}_{gi}}{\partial x_j} + \frac{\partial \tilde{U}_{gj}}{\partial x_i} \right) - \frac{1}{3}\frac{\partial \tilde{U}_{gi}}{\partial x_i}\delta_{ij} \right] + 2\mu_g \left[\frac{1}{2}\left(\frac{\partial \overline{U''_{gi}}}{\partial x_j} + \frac{\partial \overline{U''_{gj}}}{\partial x_i} \right) - \frac{1}{3}\frac{\partial \overline{U''_{gi}}}{\partial x_i}\delta_{ij} \right]$$

$$\underline{\underline{\bar{\tau}_{gij} = 2\mu_g S_{gij}\left(\tilde{\mathbf{U}}_g\right) + 2\mu_g S_{gij}\left(\overline{\mathbf{U}''_g}\right)}}$$

$$S_{gij}\left(\tilde{\mathbf{U}}_g\right) \equiv \frac{1}{2}\left(\frac{\partial \tilde{U}_{gi}}{\partial x_j} + \frac{\partial \tilde{U}_{gj}}{\partial x_i} \right) - \frac{1}{3}\frac{\partial \tilde{U}_{gi}}{\partial x_i}\delta_{ij}$$

$$S_{gij}\left(\overline{\mathbf{U}''_g}\right) \equiv \frac{1}{2}\left(\frac{\partial \overline{U''_{gi}}}{\partial x_j} + \frac{\partial \overline{U''_{gj}}}{\partial x_i} \right) - \frac{1}{3}\frac{\partial \overline{U''_{gi}}}{\partial x_i}\delta_{ij}$$

Gas Phase Momentum Equations

$$\frac{\partial}{\partial t}\left(\bar{\varepsilon}_g \rho_g \tilde{U}_{gi}\right) + \frac{\partial}{\partial x_j}\left(\bar{\varepsilon}_g \rho_g \tilde{U}_{gj} \tilde{U}_{gi}\right)$$

$$= -\bar{\varepsilon}_g \frac{\partial \bar{P}_g}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{gij} + \tau_{\tilde{T}gij}\right) - \bar{I}_{gsi} + \bar{\varepsilon}_g \rho_g g_i + \overline{\varepsilon'_g \frac{\partial P'_g}{\partial x_i}}$$

$$\underline{\underline{\bar{\tau}_{gij} = 2\mu_g S_{gij}\left(\tilde{\mathbf{U}}_g\right) + 2\mu_g S_{gij}\left(\overline{\mathbf{U}''_g}\right)}}$$

$$\begin{aligned}\bar{I}_{gsi} = {} & \bar{\hat{\beta}}_{gs}\, \bar{\varepsilon}_s \left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \bar{\hat{\beta}}_{gs} \overline{U''_{gi}} \\ & + \overline{\hat{\beta}'_{gs}\, \varepsilon'_s}\left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \bar{\varepsilon}_s \overline{\hat{\beta}'_{gs}\left(U''_{gi} - U''_{si}\right)} \\ & - \overline{\hat{\beta}'_{gs}\left(\varepsilon'_g U''_{gi} + \varepsilon'_s U''_{si}\right)}.\end{aligned}$$

Specific Gas Phase Turbulence Kinetic Energy

$$\bar{\varepsilon}_g \tilde{k}_g = \frac{1}{2}\overline{\varepsilon_g U''_{gi} U''_{gi}}$$

Total energy of gas flow

$$\frac{1}{2}\overline{\rho_g \varepsilon_g U_{gi} U_{gi}} = \frac{1}{2}\overline{\rho_g \varepsilon_g \left(U_g\right)^2} = \frac{1}{2}\overline{\rho_g \varepsilon_g \left(\tilde{\mathbf{U}}_g + \mathbf{U}''_g\right)^2}$$

$$= \frac{1}{2}\rho_g \bar{\varepsilon}_g \left(\tilde{U}_g\right)^2 + \rho_g \bar{\varepsilon}_g \tilde{k}_g$$

The first term is the energy density of the gas flow due to the <u>combined mean motion and turbulent motion correlated with the void fraction</u>.

The second term is the gas energy density due to the <u>residual turblent motion, *i.e.*, that not correlated with voidage fluctuations</u>.

Besnard, D.C., and F.H. Harlow, 1988.

$$\frac{1}{2}\overline{\rho_g \varepsilon_g \left(U_g\right)^2} = \frac{1}{2}\overline{\rho_g \varepsilon_g \left(\overline{\mathbf{U}}_g + \mathbf{U}'_g\right)^2} = ??$$

Gas Phase Turbulence Kinetic Energy

$$\underbrace{\frac{\partial}{\partial t}\left(\rho_g \bar{\varepsilon}_g \tilde{k}_g\right)}_{\substack{\text{Rate of change of} \\ \text{turbulence kinetic energy per unit mass} \\ \text{due to non-stationarity}}} + \underbrace{\frac{\partial}{\partial x_j}\left(\rho_g \bar{\varepsilon}_g \tilde{k}_g \tilde{U}_{gj}\right)}_{\substack{\text{Rate of change of} \\ \text{turbulence kinetic energy per unit mass} \\ \text{due to convection by the mean flow}}}$$

$$+ \underbrace{\rho_g \overline{\varepsilon_g U''_{gi} U''_{gj}} \frac{\partial \tilde{U}_{gi}}{\partial x_j}}_{\substack{\text{Rate of production of} \\ \text{turbulence kinetic energy} \\ \text{from the mean flow gradient}}} + \underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_g \overline{\varepsilon_g U''_{gj} U''_{gi} U''_{gi}}\right)}_{\substack{\text{Transport of} \\ \text{turbulence kinetic energy} \\ \text{due to the turbulence itself}}}$$

$$= \underbrace{- \bar{\varepsilon}_g \overline{U''_{gi} \frac{\partial P'_g}{\partial x_i}} - \overline{\varepsilon'_g U''_{gi} \frac{\partial P'_g}{\partial x_i}}}_{\substack{\text{Transport of} \\ \text{turbulence kinetic energy} \\ \text{due to the pressure fluctuations}}} + \underbrace{\overline{\varepsilon_g U''_{gi} \frac{\partial \tau_{gij}}{\partial x_j}}}_{\text{Dissipation}} \underbrace{- \overline{U''_{gi} I_{gsi}}}_{\substack{\text{Rate of transfer of} \\ \text{(gas phase) turbulence kinetic energy} \\ \text{to the granular phase}}}$$

Correlations

Volume fraction fluctuation - velocity fluctuation correlations

$$\overline{\varepsilon'_g U'_{gi}} \text{ and } \overline{\varepsilon'_s U'_{si}}$$

Specific gas phase turbulence Reynolds/Favre stresses

$$\bar{\varepsilon}_g \, \tau_{\tilde{T}gij} \equiv -\overline{\varepsilon_g \, U''_{gi} \, U''_{gj}} \text{ and } \bar{\varepsilon}_s \, \tau_{\tilde{T}sij} \equiv -\overline{\varepsilon_s \, U''_{si} \, U''_{sj}}$$

Drag related correlations

$$\bar{\hat{\beta}}_{gs}, \ \overline{U''_{gi}} = -\tilde{U}'_{gi},$$

$$\overline{\varepsilon'_s \hat{\beta}'_{gs}} = -\overline{\varepsilon'_g \hat{\beta}'_{gs}}, \ \overline{\hat{\beta}'_{gs}\left(U''_{gi} - U''_{si}\right)}, \text{ and } \overline{\hat{\beta}'_{gs} \, \varepsilon'_s \left(U''_{gi} - U''_{si}\right)}$$

Stress related correlation $\overline{\varepsilon'_g \dfrac{\partial P'_g}{\partial x_j}}, \ \overline{U''_{gi}}$

Closures
velocity & volume fraction

Using an eddy - viscosity type turbulence model

$$\overline{\varepsilon_g' U_{gi}'} \approx -\frac{\nu_{tg}}{\sigma_{\varepsilon g}} \nabla \bar{\varepsilon}_g \text{ and } \overline{\varepsilon_s' U_{si}'} \approx -\frac{\nu_{ts}}{\sigma_{\varepsilon s}} \nabla \bar{\varepsilon}_s = +\frac{\nu_{ts}}{\sigma_{\varepsilon s}} \nabla \bar{\varepsilon}_g$$

These correlations are proportional, but opposite in sign

$$\overline{\varepsilon_s' U_{si}'} = -\left(\frac{\nu_{ts}}{\nu_{tg}} \frac{\sigma_{\varepsilon g}}{\sigma_{\varepsilon s}} \right) \overline{\varepsilon_g' U_{gi}'}.$$

$\nu_{tg}\left(\nu_{ts}\right)$ - gas (solids) phase kinematic eddy viscosity

$\sigma_{\varepsilon g}\left(\sigma_{\varepsilon s}\right)$ - turbulent Prandtl number for gas (solids)

Closures
velocity relations

Using the identities

$$\tilde{U}_{gi} = \overline{U}_{gi} + \overline{\varepsilon_g' U_{gi}'} \big/ \overline{\varepsilon}_g \text{ and } \tilde{U}_{si} = \overline{U}_{si} + \overline{\varepsilon_s' U_{si}'} \big/ \overline{\varepsilon}_s ,$$

$$\tilde{U}_{gi} - \overline{U}_{gi} \approx -\frac{\nu_{tg}}{\sigma_{\varepsilon g}\overline{\varepsilon}_g}\nabla\overline{\varepsilon}_g$$

$$\tilde{U}_{si} - \overline{U}_{si} \approx -\frac{\nu_{ts}}{\sigma_{\varepsilon s}\overline{\varepsilon}_s}\nabla\overline{\varepsilon}_g = +\frac{\nu_{ts}}{\sigma_{\varepsilon s}\left(1-\overline{\varepsilon}_g\right)}\nabla\overline{\varepsilon}_g$$

Closures phase interaction

Retaining lowest order terms,

$$\bar{I}_{gsi} \approx \overline{\hat{\beta}}_{gs}\,\bar{\varepsilon}_s\left(\tilde{U}_{gi} - \tilde{U}_{si}\right) + \overline{\hat{\beta}}_{gs}\overline{U''_{gi}}$$

Using an eddy - viscosity type turbulence model

$$\overline{U''_{gi}} = -\overline{\varepsilon'_g U'_{gi}} \Big/ \bar{\varepsilon}_g \approx \frac{\nu_{tg}}{\sigma_{\varepsilon g}\bar{\varepsilon}_g}\nabla\bar{\varepsilon}_g$$

Closures: phase interaction

$\bar{\hat{\beta}}_{gs}$ has no analogue in single phase turbulence.

$$\text{N.B.,}\ \bar{\hat{\beta}}_{gs}(\mathbf{x}) \neq \hat{\beta}_{gs}\left(\bar{\varepsilon}_g(\mathbf{x}), \left|\bar{\mathbf{U}}_g(\mathbf{x}) - \bar{\mathbf{U}}_s(\mathbf{x})\right|\right)$$

Try :

$$\bar{\hat{\beta}}_{gs}(\mathbf{x}) \approx \bar{\hat{\beta}}_{gs}^{(0)}$$

$$\approx \bar{\hat{\beta}}_{gs}^{(1)} \equiv \hat{\beta}_{gs}\left(\bar{\varepsilon}_g(\mathbf{x}), \left|\tilde{\mathbf{U}}_g(\mathbf{x}) - \tilde{\mathbf{U}}_s(\mathbf{x})\right|\right)$$

where $\tilde{\mathbf{U}}_g(\mathbf{x}) - \tilde{\mathbf{U}}_s(\mathbf{x})$

$$= \bar{\mathbf{U}}_g(\mathbf{x}) - \bar{\mathbf{U}}_s(\mathbf{x}) + \overline{\varepsilon_g' U_{gi}'} \big/ \bar{\varepsilon}_g - \overline{\varepsilon_s' U_{si}'} \big/ \bar{\varepsilon}_s$$

To Do List

- Add granular temperature equation
  - Indentify closures
- Extract dissipation, ϵ
- Closures for k & ϵ
- Look at energy cascade
- Equation for the full stress tensor, τ
- Compare in detail with Reynolds formalism
- Formulate drag terms for a chosen form

- Develop closure relationships

  … related to specific experimental data

  (A NEVER ENDING TASK)


Thank you
Questions?