Stochastic modeling of drag forces in Euler-Lagrange simulations of particle-laden flows

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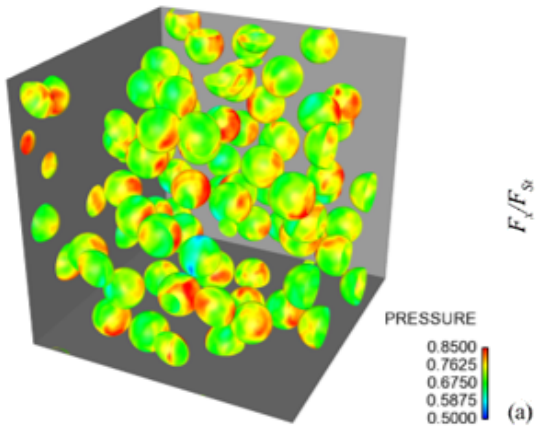
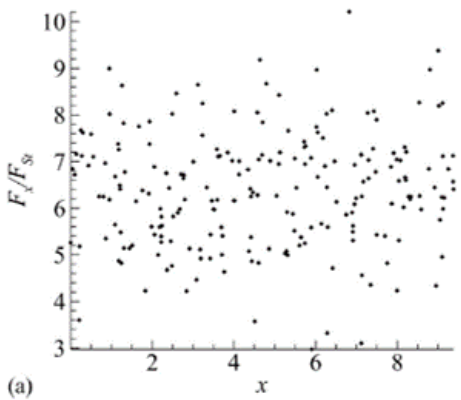
Drag force(s) acting on a collection of particles

Suspensions exhibit significant drag variation



Shallcross *et al.* (IP)



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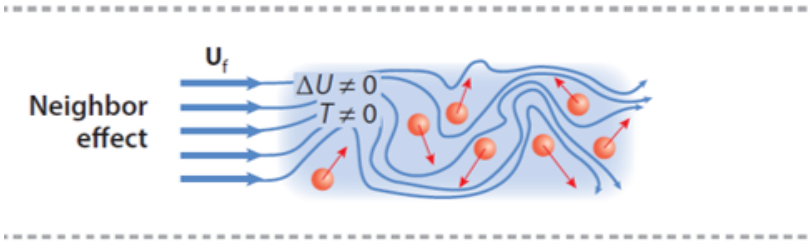

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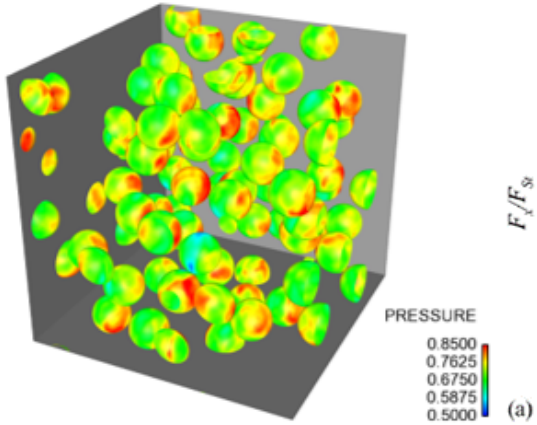
Existing drag laws fail to capture higher-order (HO) statistics [1–3]

- Particle velocity variance
- Particle dispersion



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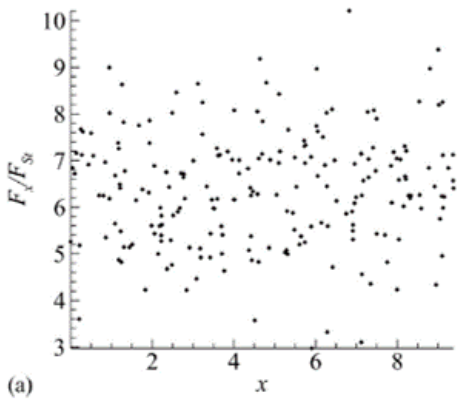

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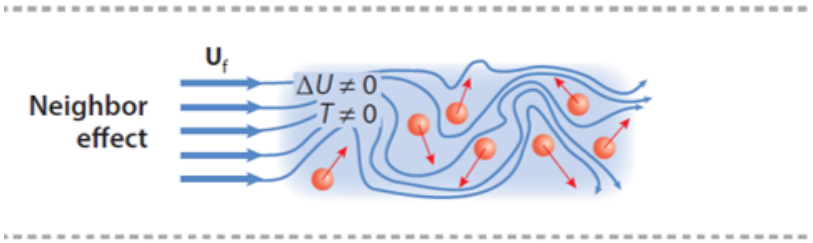

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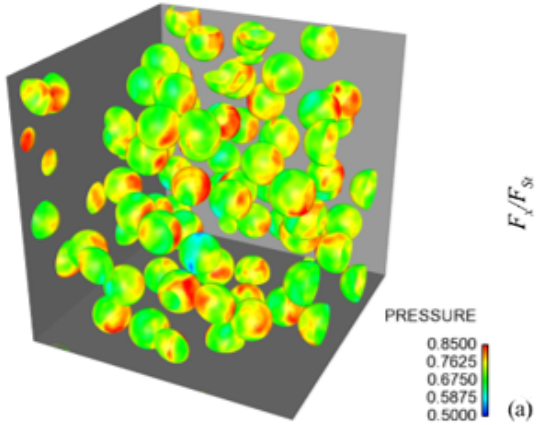
Why?



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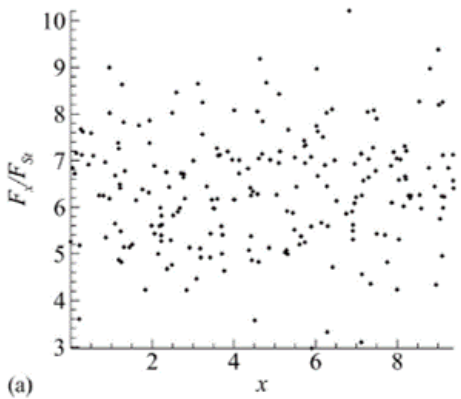

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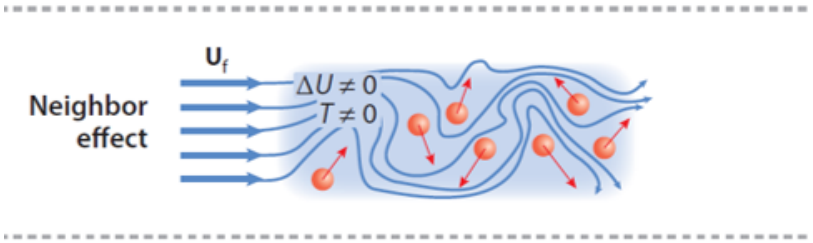

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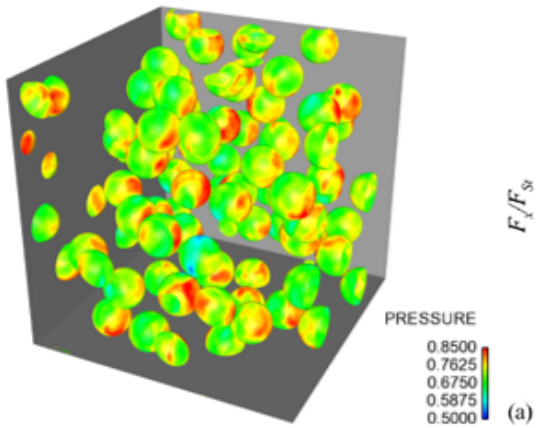

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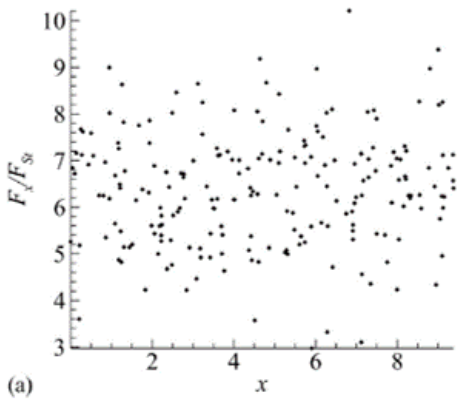

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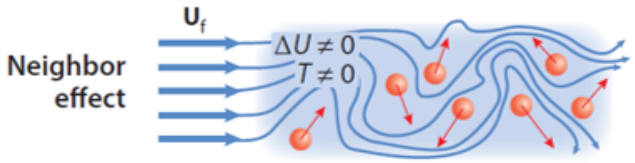
Why?

They coarse-grain the microstructure...



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Suspensions exhibit

Hydrodynamic forces:

$$m_p^{(i)} \frac{\mathrm{d}\boldsymbol{U}_p^{(i)}}{\mathrm{d}t} = V_p^{(i)} \nabla \cdot \bar{\boldsymbol{\tau}} \left[\boldsymbol{X}_p^{(i)} \right] + \int \boldsymbol{\tau}' \cdot \boldsymbol{n} \, \mathrm{d}S$$

Isolated sphere (Maxey-Riley)

$$\underbrace{3\pi\mu_f d_p f_{\mathrm{iso}} \left(\boldsymbol{u}_f - \boldsymbol{U}_p^{(i)} \right)}_{\text{Quasi-steady drag}} + \underbrace{\frac{\rho_f V_p^{(i)}}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{u}_f - \boldsymbol{U}_p^{(i)} \right)}_{\text{Added mass}} + \underbrace{\frac{3}{2} \sqrt{\pi \rho_f \mu_f} d_p^2 \int_0^t \left[\frac{\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\boldsymbol{u}_f - \boldsymbol{U}_p^{(i)} \right)}{(t-\tau)^{1/2}} \right] \mathrm{d}\tau}_{\text{Basset history}}$$

We seek a stochastic framework that incorporates these effects in the drag statistics of a suspension

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Talk forecast

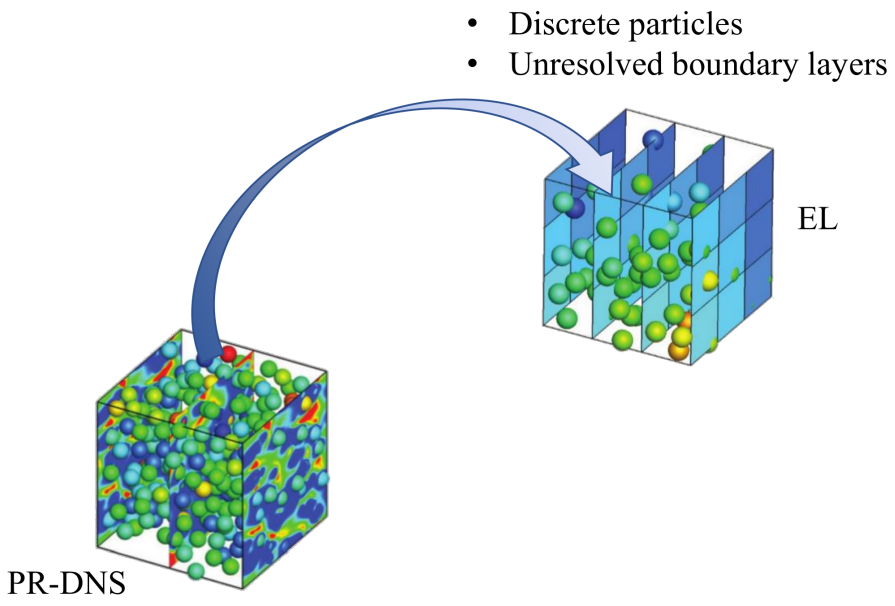
Emphasize stochastic EL framework

- Hydrodynamic forces
- HO particle moments
- Statistical approach

❶ **Stochastic hierarchy** [1]
  - ▶ Langevin Eqs.

❷ **Stochastic EL solver** [2]
  - ▶ Improved predictions

❸ **Closure for EE solvers** [3]
  - ▶ Hydrodynamic sources



[1] Lattanzi *et al.* (2020)
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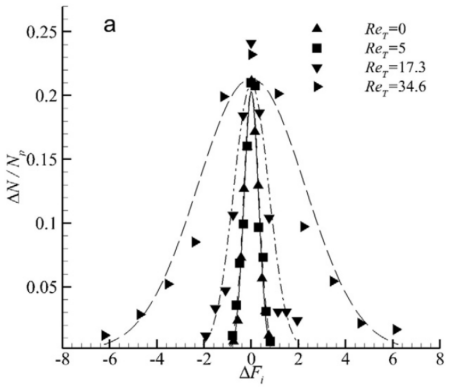
Neighbor-induced drag statistics

PR-DNS studies show Gaussian PDFs [1–3]



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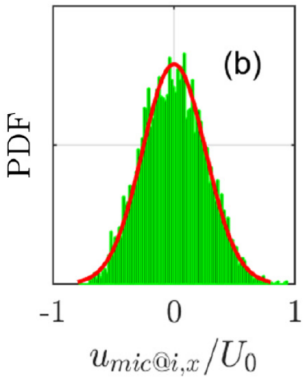
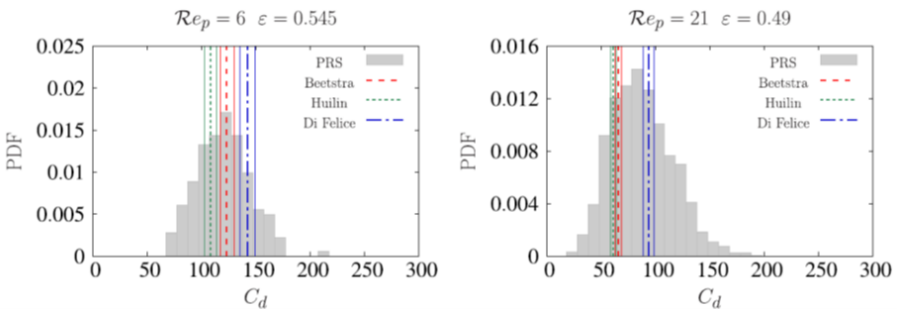


Balachandar (2020)



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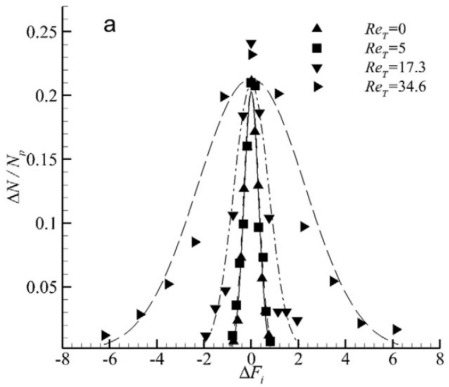
Expand unresolved drag about the mean

$$\int \boldsymbol{\tau}' \cdot \boldsymbol{n}\, \mathrm{d}S = \langle \boldsymbol{F}_d \rangle + \boldsymbol{F}_d''^{(i)}$$



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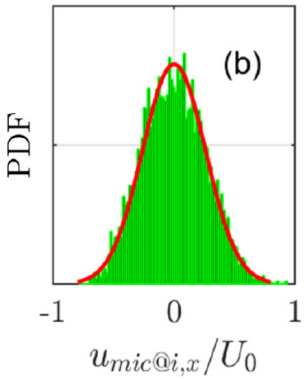

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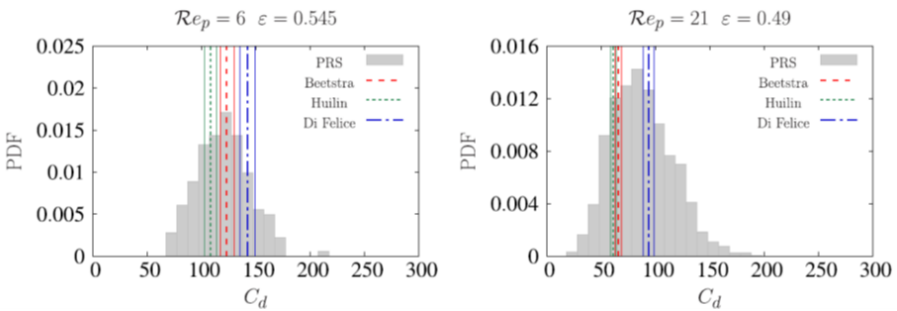

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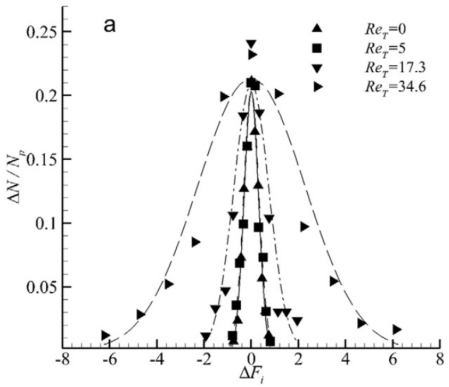
- Specify statistics via $\boldsymbol{F}_d''^{(i)}$
  - Deterministic PIEP [1]
  - Stochastic Langevin [3]



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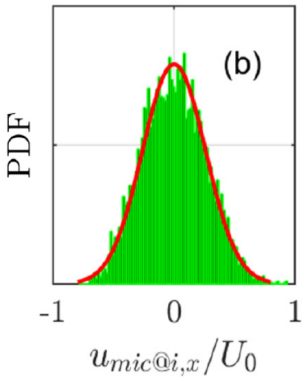

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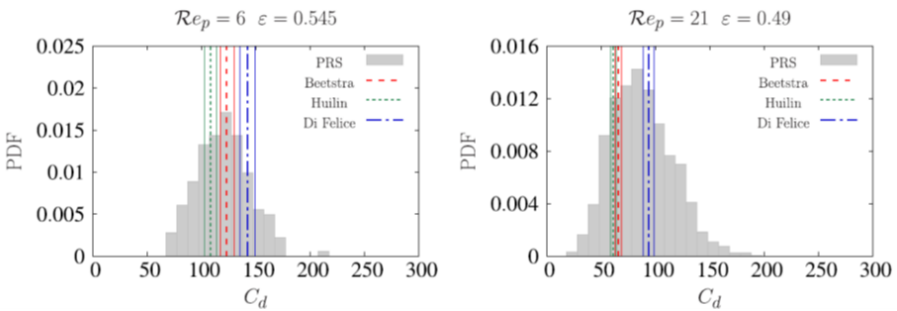

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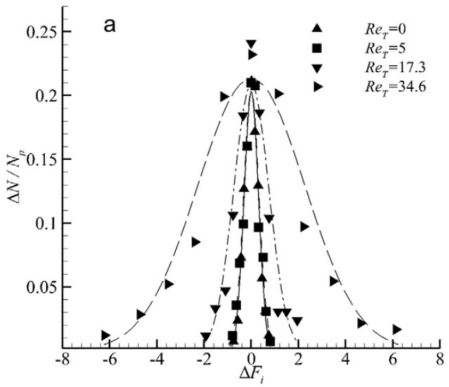
Choose $\langle \boldsymbol{F}_d \rangle$ correlation

- $\langle \boldsymbol{F}_d \rangle = f\left(\mathrm{Re}_p, \phi\right)$
- $\langle \boldsymbol{F}_d \rangle = f\left(\mathrm{Re}_p, \phi, \rho_p/\rho_f\right)$



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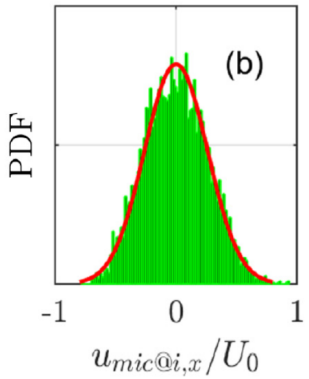

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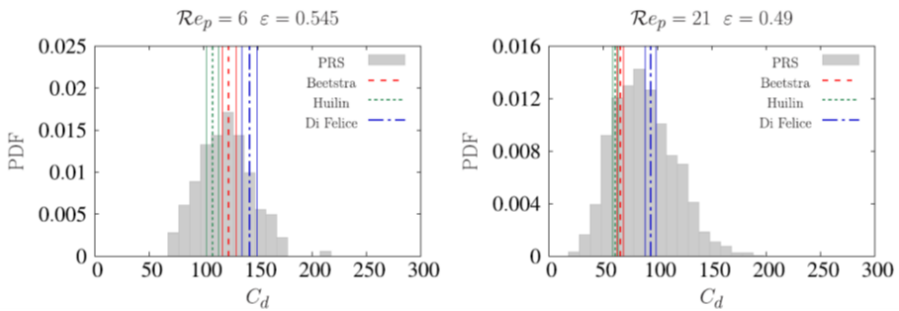

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Consequence of force Langevin equation

$$\mathrm{d}\boldsymbol{F}_d''^{(i)} = -\frac{1}{\tau_F}\boldsymbol{F}_d''^{(i)}\mathrm{d}t + \frac{\sqrt{2}\sigma_F}{\sqrt{\tau_F}}\mathrm{d}\boldsymbol{W}_t$$

Evolution of particle-phase moments

- Gaussian drag fluctuations
- Exponential ACF

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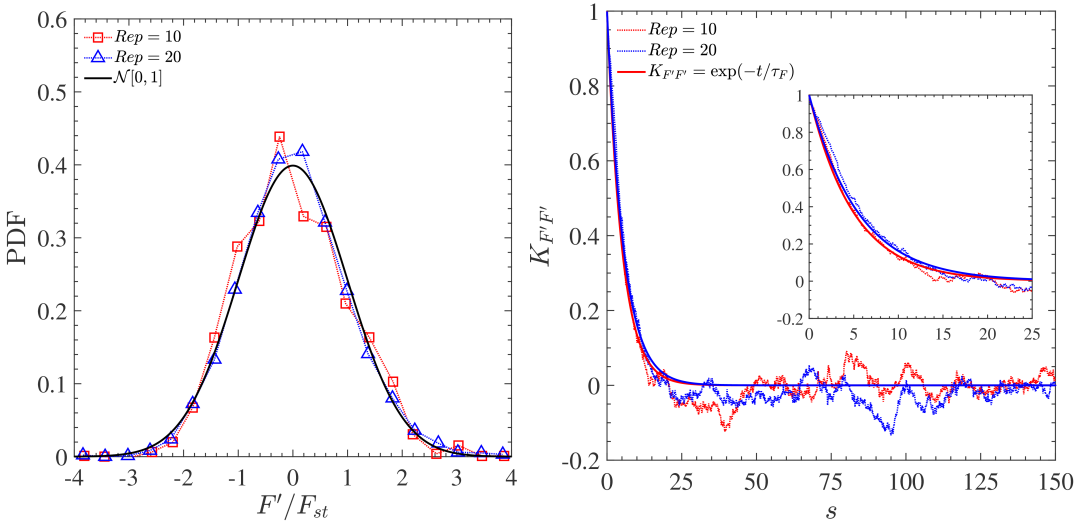

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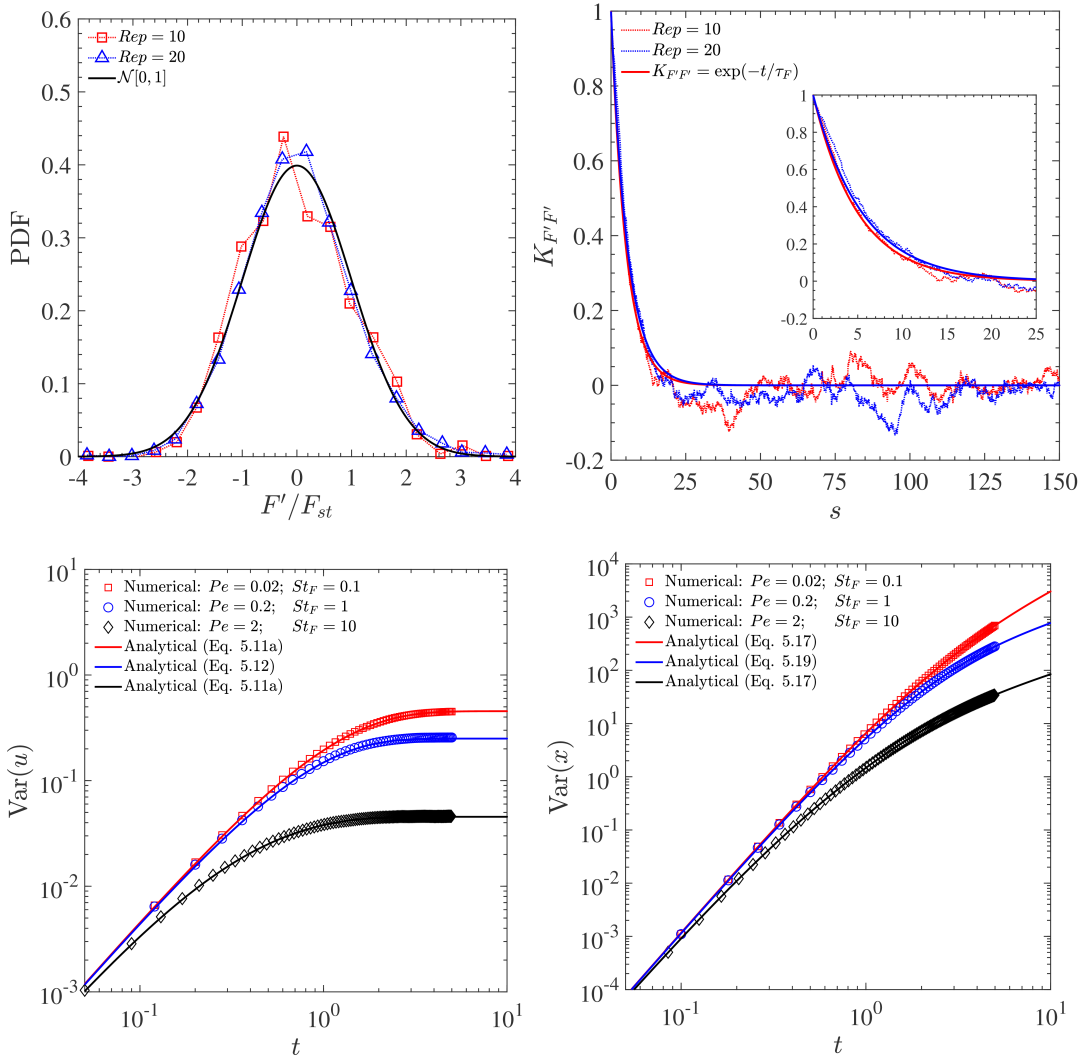

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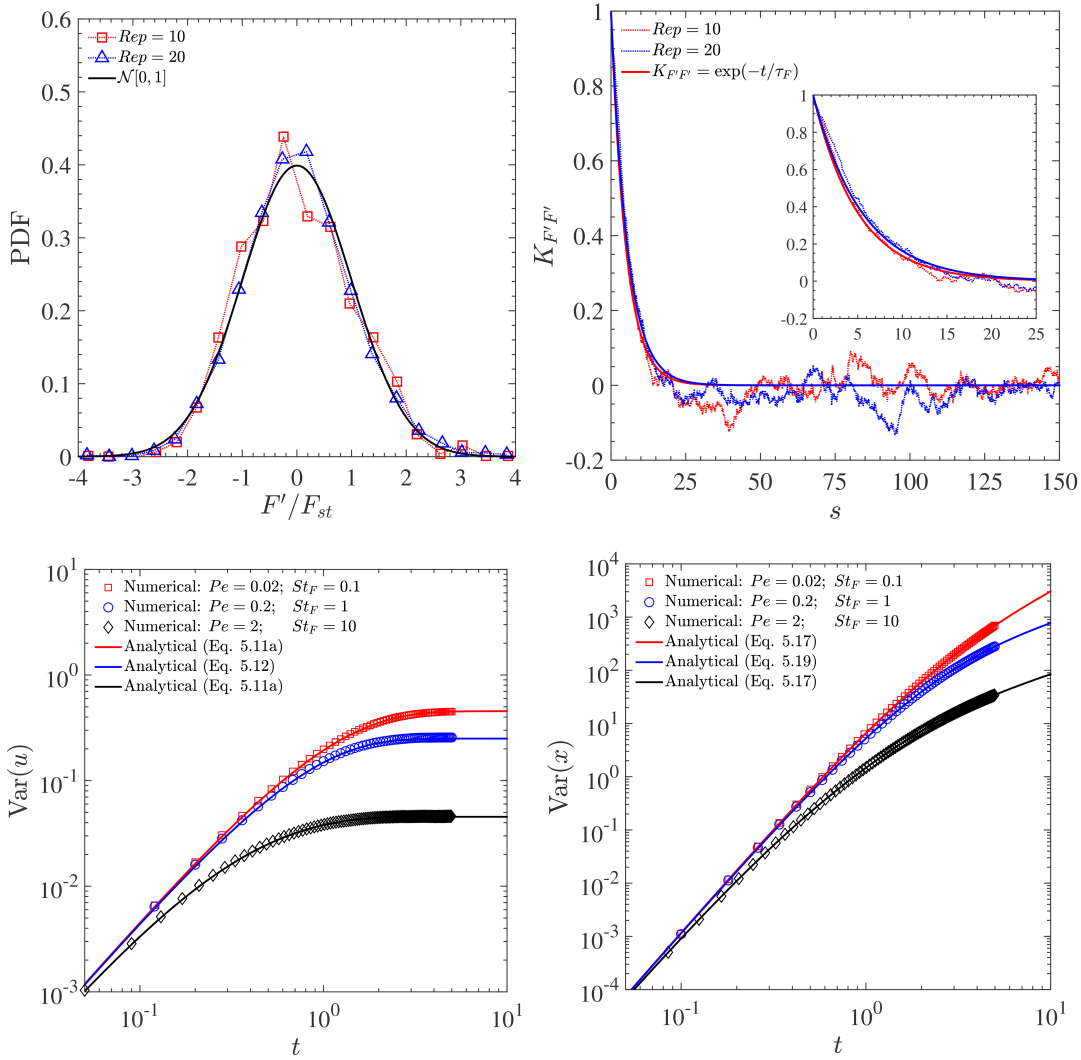

Higher-order drag statistics consistent with PR-DNS

Towards a stochastic drag framework

Equation of motion:

$$m_p^{(i)} \frac{\mathrm{d}\boldsymbol{U}_p^{(i)}}{\mathrm{d}t} = \sum_{j=1}^{N} \boldsymbol{F}_{\mathrm{col}}^{(ij)} + V_p^{(i)} \nabla \cdot \bar{\boldsymbol{\tau}} \left[\boldsymbol{X}_p^{(i)} \right] + \int \boldsymbol{\tau}' \cdot \boldsymbol{n} \, \mathrm{d}S$$

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$$\left\langle \boldsymbol{F}_d^{(i)} \right\rangle = f\left(\mathrm{Re}_p, \phi\right) \checkmark$$

Tenneti *et al.* (2011)

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Chapman & Cowling (1970)

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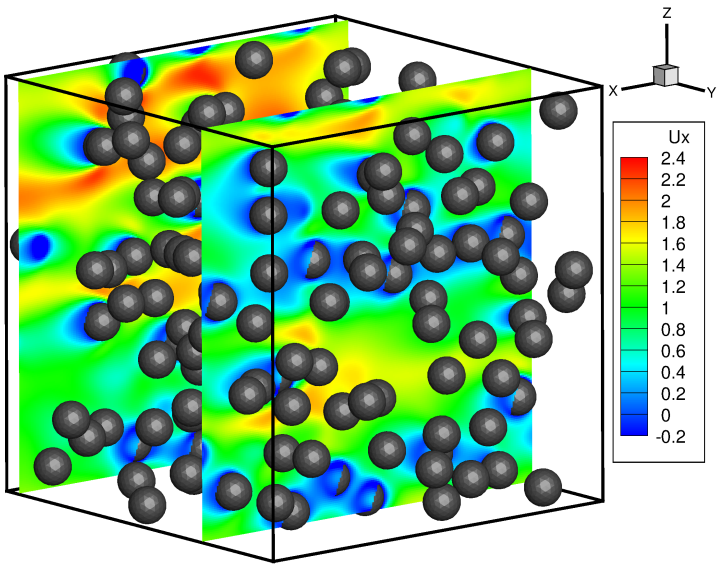
Chapman & Cowling (1970)

Need model for σ_F

A correlation for drag variance

PR-DNS of fixed assemblies

- PUReIBM



Tavanashad & Subramaniam (2020)

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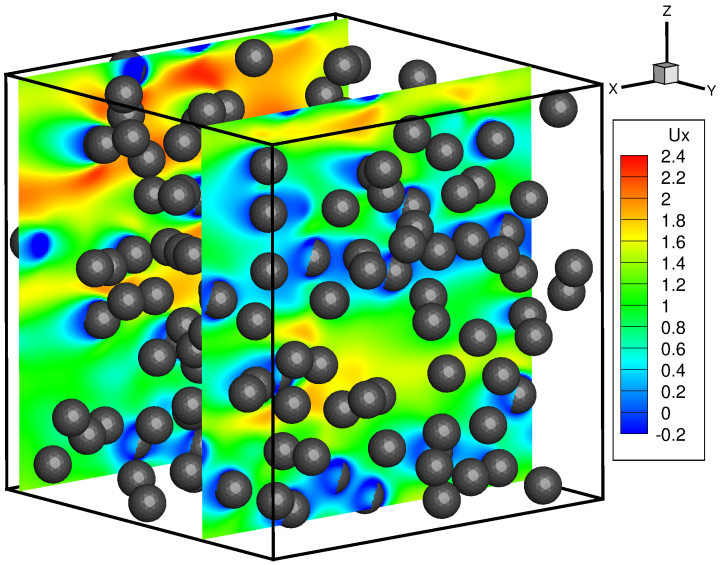

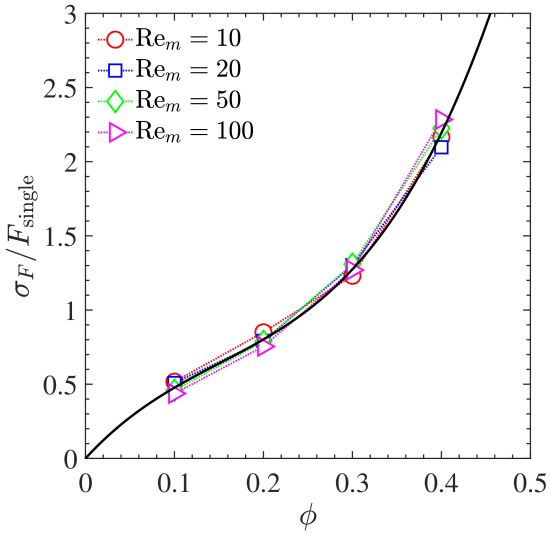

Tavanashad & Subramaniam (2020)

$$f_{\phi}^{\sigma_F} = 6.52\phi - 22.56\phi^2 + 49.90\phi^3$$

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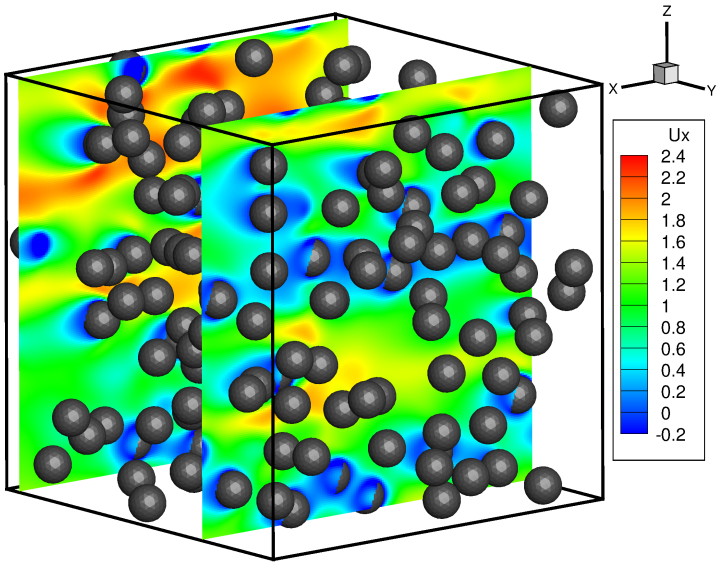
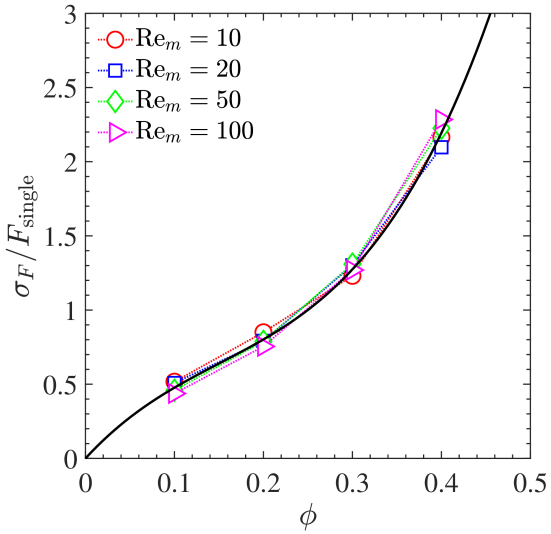




Tavanashad & Subramaniam (2020)

$$\frac{\sigma_F}{m_p^{(i)}} \equiv f_\phi^{\sigma_F} \frac{F_{\mathrm{single}}}{m_p^{(i)}} = f_\phi^{\sigma_F} f_{\mathrm{iso}} \frac{(1-\phi)\left\| \boldsymbol{u}_f\left[\boldsymbol{X}_p^{(i)}\right] - \boldsymbol{U}_p^{(i)} \right\|}{\tau_p}$$

$$f_\phi^{\sigma_F} = 6.52\phi - 22.56\phi^2 + 49.90\phi^3$$

$$f_{\mathrm{iso}} = \left(1 + 0.15\mathrm{Re}_p^{0.687}\right)$$

EL framework

NGA low mach solver

- Volume filtering, 2nd order scheme

$$\frac{\partial}{\partial t}\left((1-\phi)\rho_f\right) + \nabla\cdot\left((1-\phi)\rho_f \boldsymbol{u}_f\right) = 0$$

$$\frac{\partial}{\partial t}\left((1-\phi)\rho_f \boldsymbol{u}_f\right) + \nabla\cdot\left((1-\phi)\rho_f \boldsymbol{u}_f \otimes \boldsymbol{u}_f\right) = \nabla\cdot\bar{\boldsymbol{\tau}} + (1-\phi)\rho_f \boldsymbol{g} - \boldsymbol{\mathcal{F}}_{\text{inter}} + \boldsymbol{\mathcal{F}}_{\text{mfr}}$$

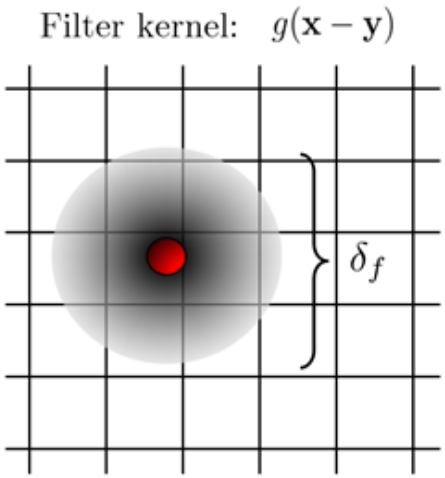
Lagrangian particle tracking

- Soft-sphere (multiple, enduring contacts)

$$\frac{\mathrm{d}\boldsymbol{X}_p^{(i)}}{\mathrm{d}t} = \boldsymbol{U}_p^{(i)}$$

$$m_p^{(i)}\frac{\mathrm{d}\boldsymbol{U}_p^{(i)}}{\mathrm{d}t} = \sum_{j=1}^{N} \boldsymbol{F}_{\text{col}}^{(ij)} + \boldsymbol{F}_{\text{inter}}^{(i)} + m_p^{(i)}\boldsymbol{g}$$

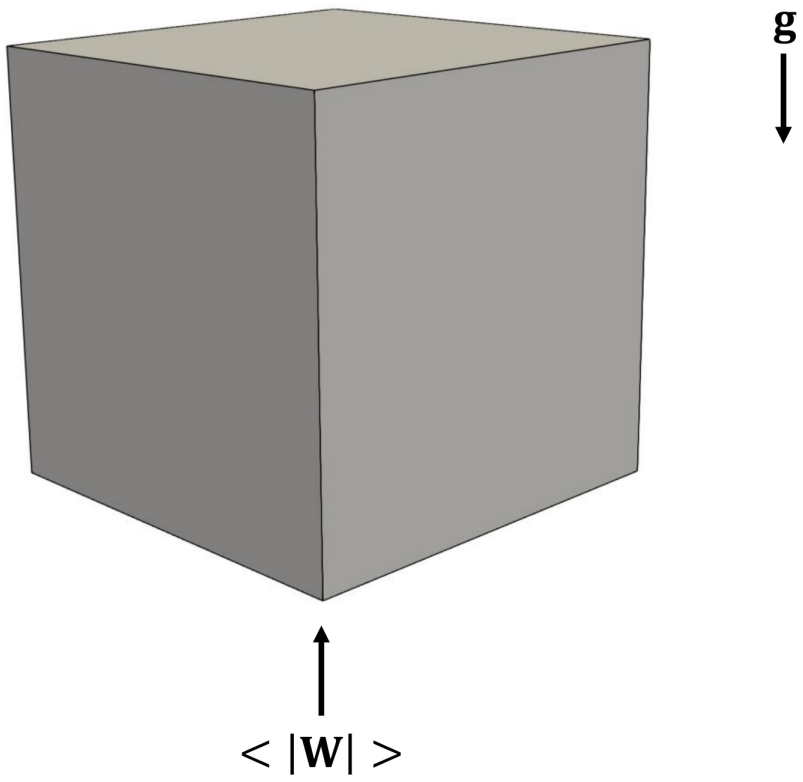
$$\boldsymbol{F}_{\text{inter}}^{(i)} = V_p^{(i)}\nabla\cdot\bar{\boldsymbol{\tau}}\left[\boldsymbol{X}_p^{(i)}\right] + \left\langle \boldsymbol{F}_d^{(i)} \right\rangle + \boldsymbol{F}_d''^{(i)}$$



Homogeneous fluidization of elastic particles

Triply periodic box [1–2]

- Force flow rate $< |\boldsymbol{W}| >$
- Gravity opposes flow $\boldsymbol{g}$
- ρ_p/ρ_f, Re_m, ϕ



[1] Tenneti *et al.* (2016)

[2] Tavanashad *et al.* (2020)

Homogeneous fluidization of elastic particles

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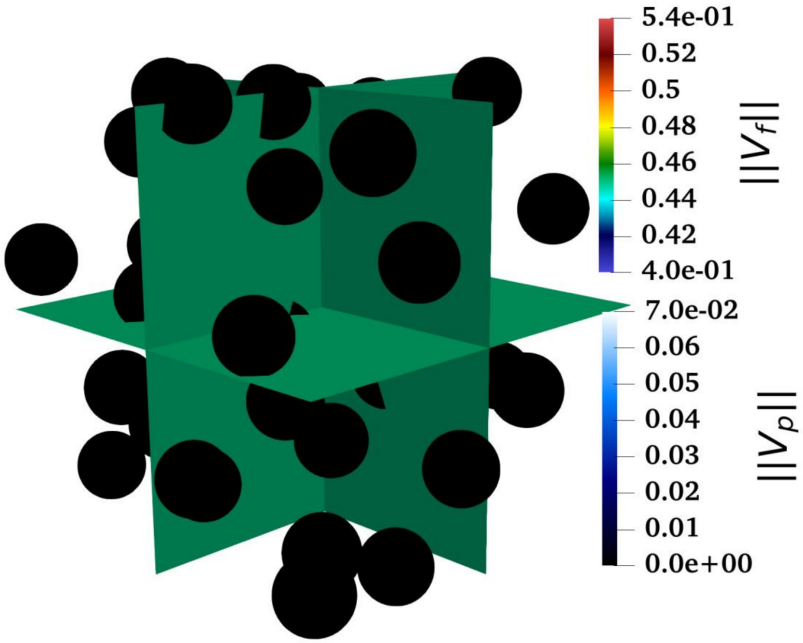
Fluidized homogeneous heating system (FHHS)

- Velocity IC $\delta(\boldsymbol{u})$



[1] Tenneti *et al.* (2016)

[2] Tavanashad *et al.* (2020)

Homogeneous fluidization of elastic particles

Triply periodic box [1–2]

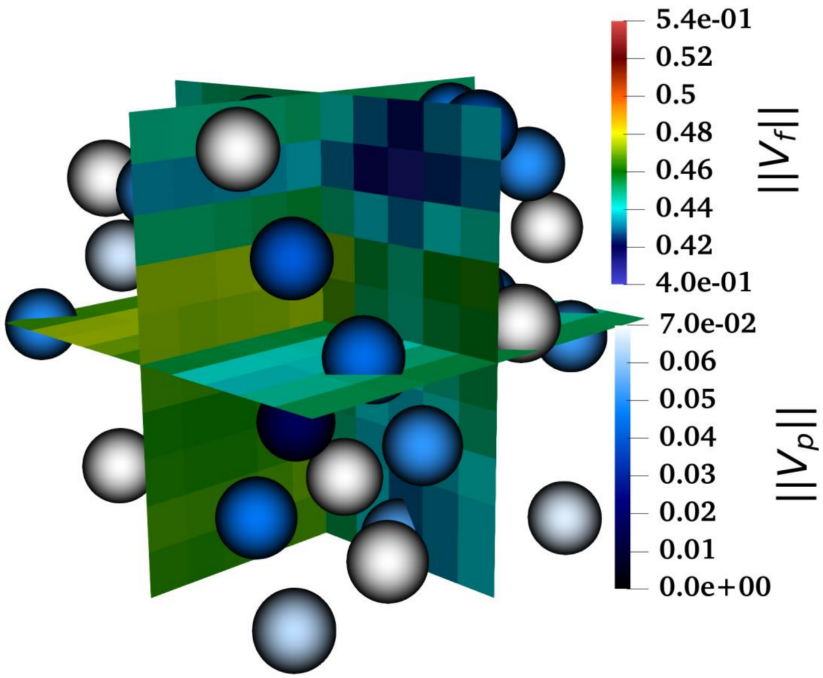
- Force flow rate $< |\boldsymbol{W}| >$
- Gravity opposes flow $\boldsymbol{g}$
- $\rho_p/\rho_f,\ \mathrm{Re}_m,\ \phi$

Fluidized homogeneous heating system (FHHS)

- Velocity IC $\delta\left(\boldsymbol{u}\right)$

Fluidized homogeneous cooling system (FHCS)

- Over-prescrived variance $\mathcal{N}[0,\ \sigma_{v,0}]$

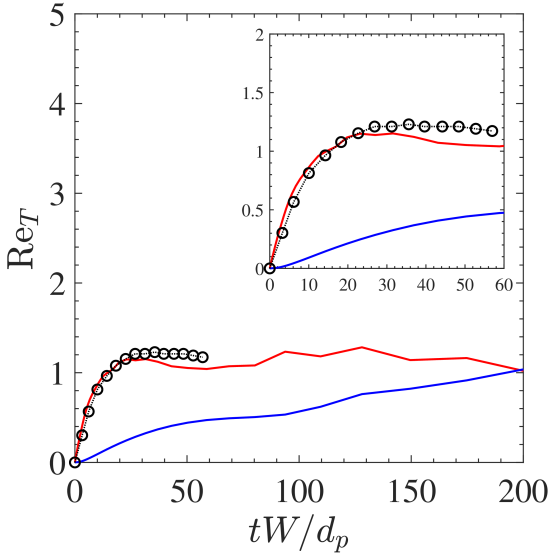


[1] Tenneti *et al.* (2016)
[2] Tavanashad *et al.* (2020)

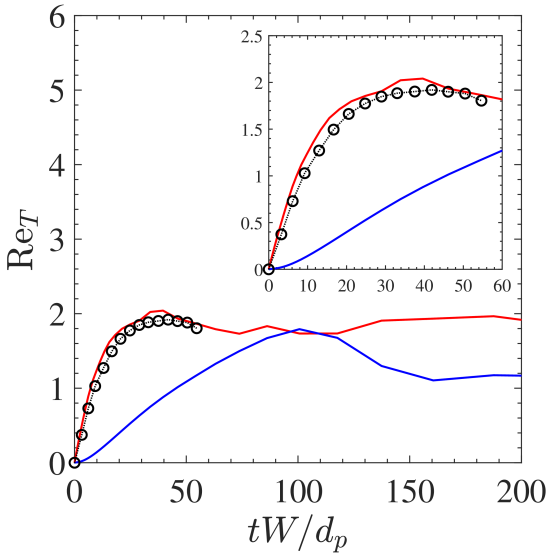
FHHS: Re_m sweep

Fixed conditions: $\rho_p/\rho_f = 100,\ \phi = 0.1$

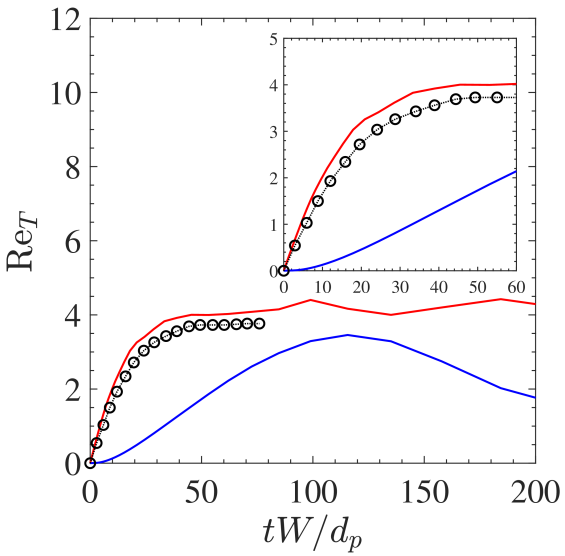
- Stochastic EL (—) Standard EL (—) PR-DNS (○)



$\mathrm{Re}_m = 10$



$\mathrm{Re}_m = 20$



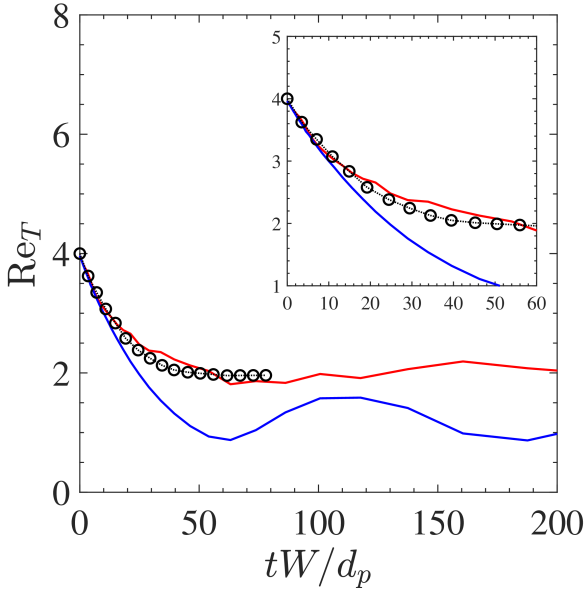
$\mathrm{Re}_m = 50$

Stochastic FL captures growth and steady velocity variance

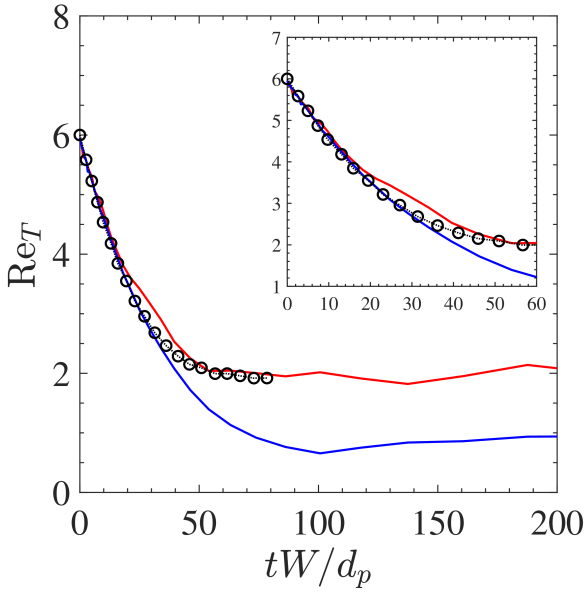
FHCS: $\mathrm{Re}_{T,0}$ sweep

Fixed conditions: $\mathrm{Re}_m = 20,\ \rho_p/\rho_f = 100,\ \phi = 0.1$

- Stochastic EL (—) Standard EL (—) PR-DNS (○)



$\mathrm{Re}_{T,0} = 4$



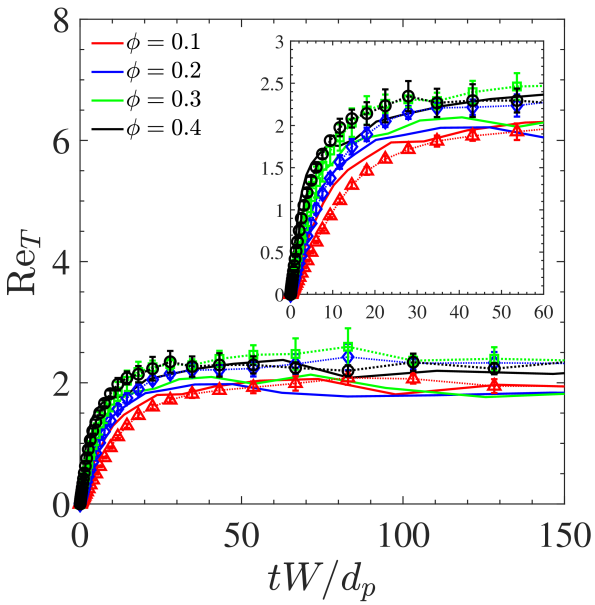
$\mathrm{Re}_{T,0} = 6$

Stochastic FL captures decay and steady velocity variance

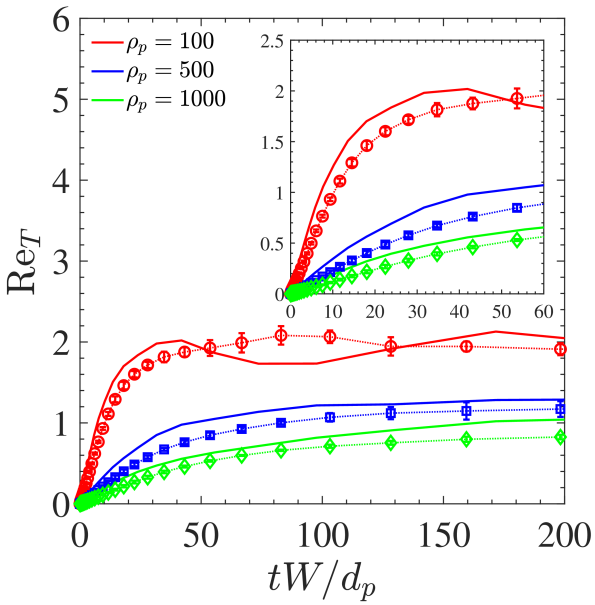
FHHS: ϕ, ρ_p/ρ_f sweep

Fixed conditions: $\mathrm{Re}_m = 20$

- Stochastic EL (—) PR-DNS (○)



$\rho_p/\rho_f = 100$



$\phi = 0.1$

Stochastic FL captures dynamics over wide range

Talk forecast: extensions

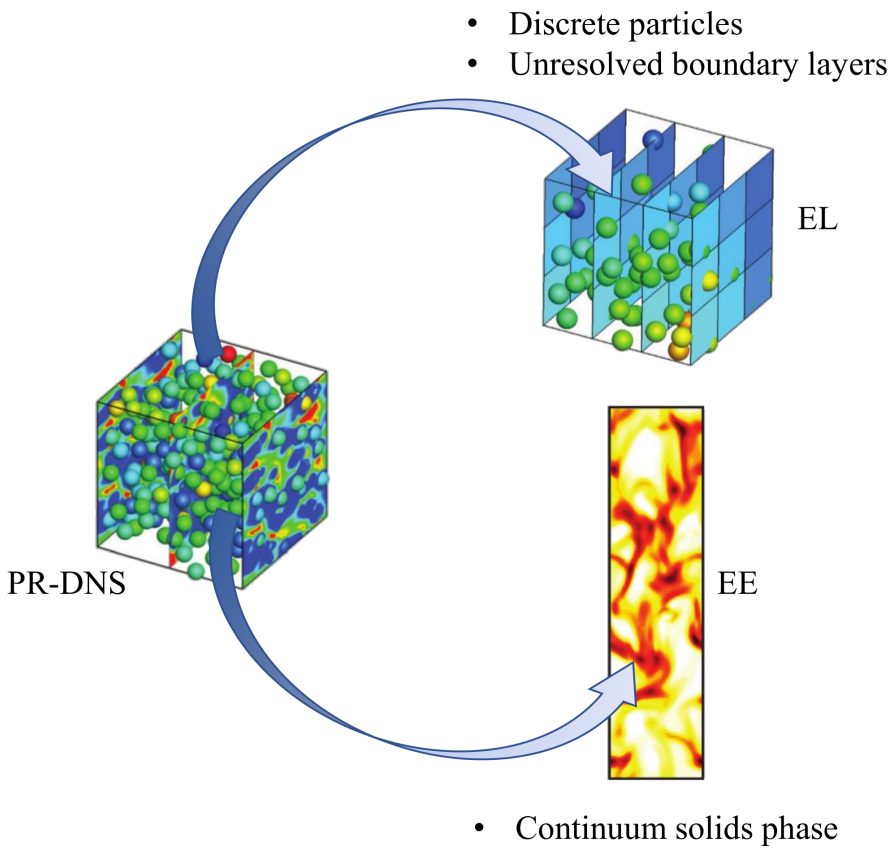
Emphasize stochastic EE framework

- Hydrodynamic forces
- HO particle moments
- Statistical approach

❶ Stochastic hierarchy [1]
- ▶ Langevin Eqs.

❷ Stochastic EL solver [2]
- ▶ Improved predictions

❸ **Closure for EE solvers** [3]
- ▶ Hydrodynamic sources



[1] Lattanzi *et al.* (2020)
[2] Lattanzi *et al.* (2021)
[3] Lattanzi *et al.* (IP)

Extension to Euler-Euler frameworks

Homogeneous, smooth, elastic spheres

$$\frac{\mathrm{d}T}{\mathrm{d}t} \equiv S - \Gamma = \frac{2}{3}\left\langle A_i' V_i' \right\rangle \qquad T = \frac{1}{3}\mathrm{Tr}\left(\left\langle \boldsymbol{V}_p' \otimes \boldsymbol{V}_p' \right\rangle\right)$$

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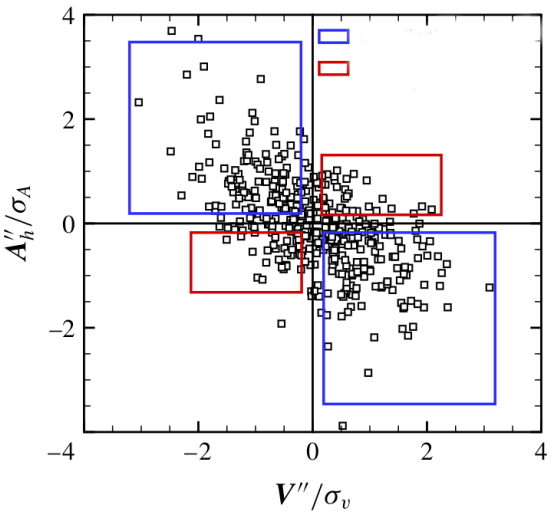

Tenneti *et al.* (2016)

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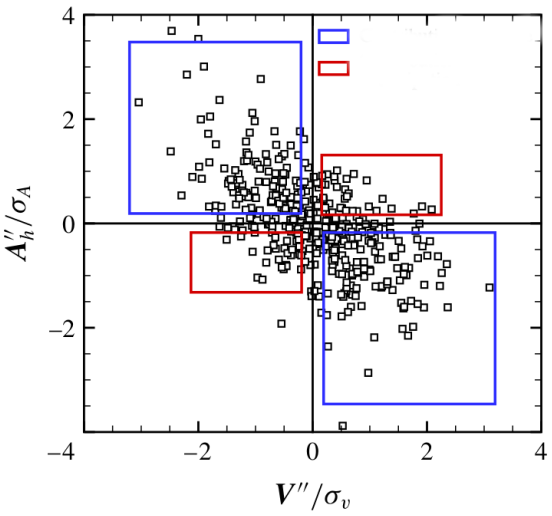
Require solution to Fokker-Planck

$$\frac{\partial P\left(v', a''; t\right)}{\partial t} + \frac{\partial}{\partial v'}\left[\left(a'' - \frac{1}{\tau_d} v'\right) P\right] - \frac{1}{\tau_{a''}} \frac{\partial}{\partial a''}\left(a'' P\right) = \frac{\sigma_{a''}^2}{\tau_a''} \frac{\partial^2 P}{\partial a''^2}.$$



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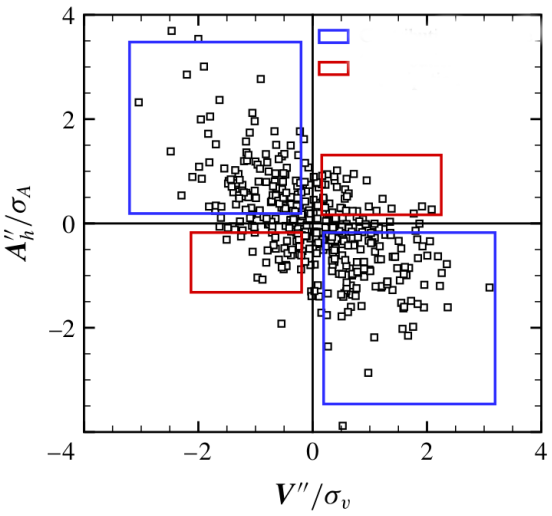

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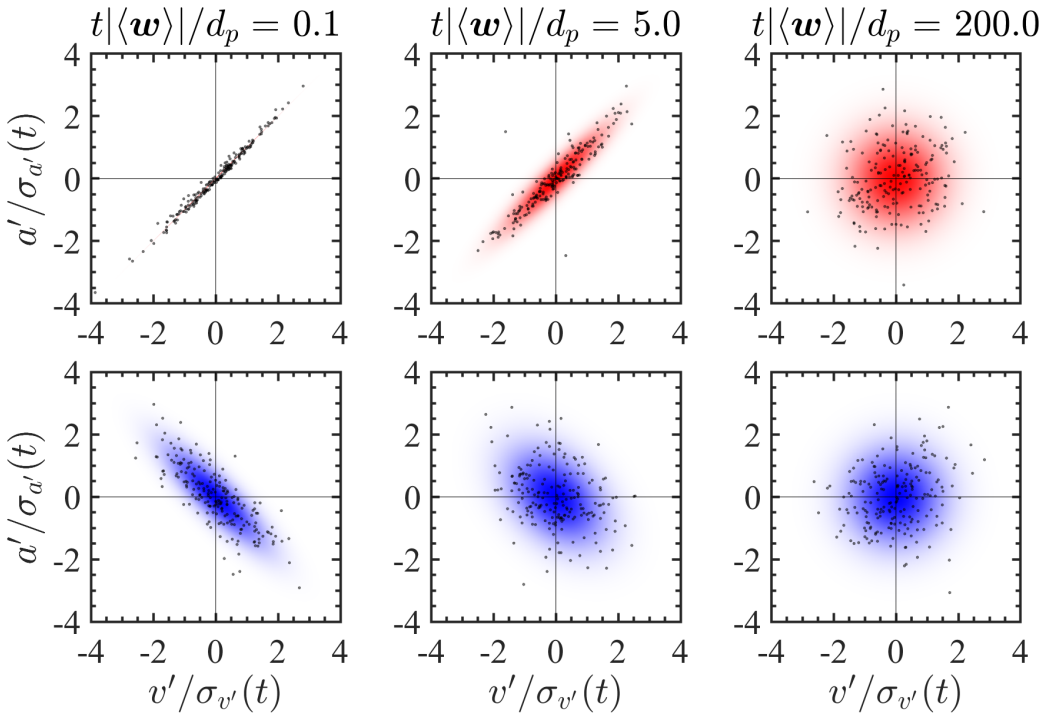
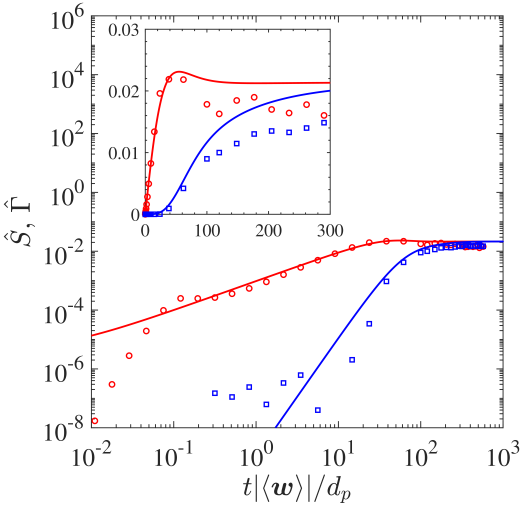
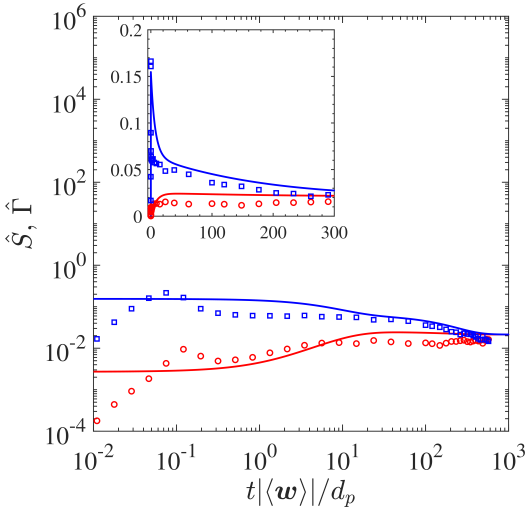


Extension to Euler-Euler frameworks

HHS S & Γ



HCS S & Γ



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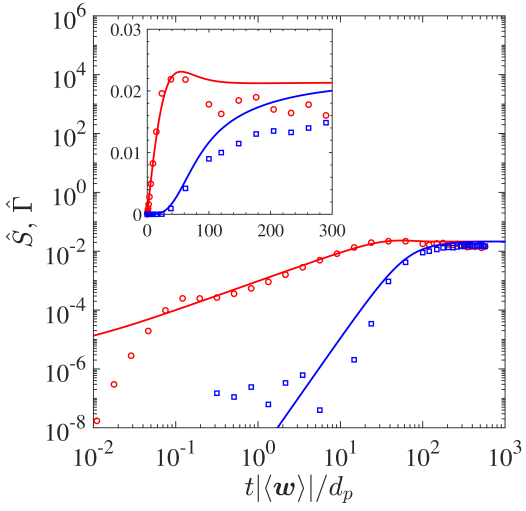

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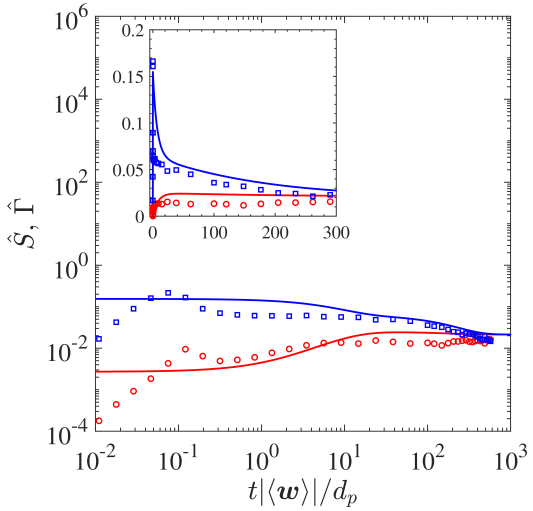
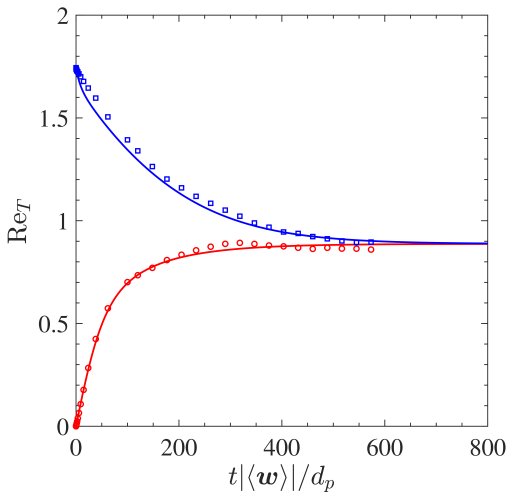


$T(t)$



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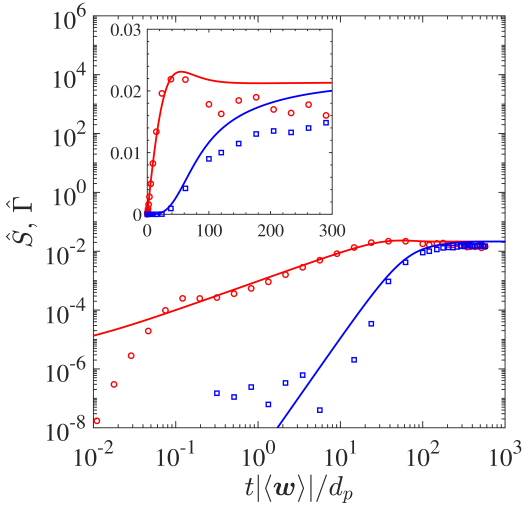

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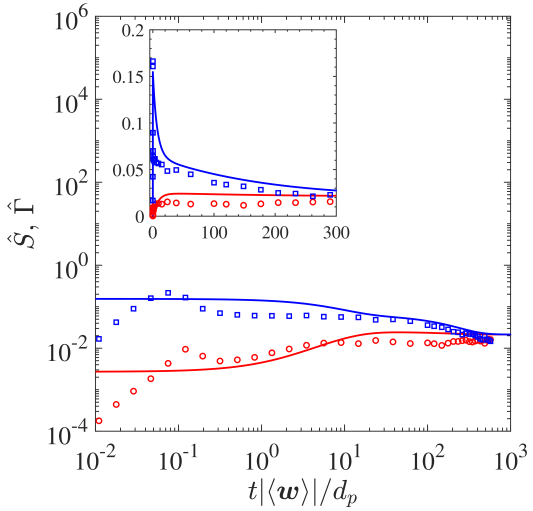

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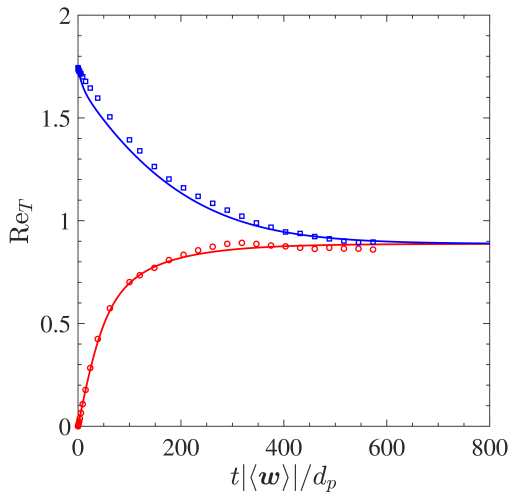
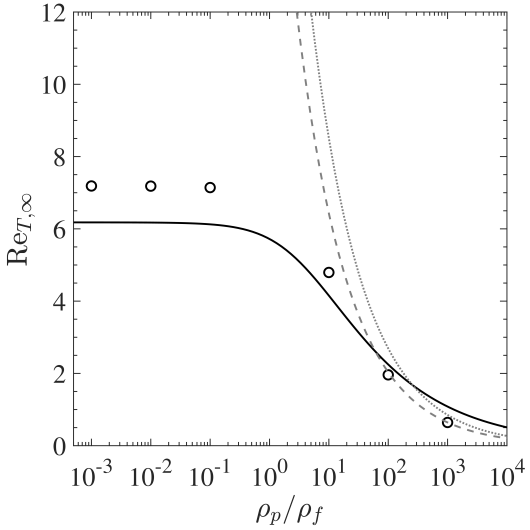


Density ratio



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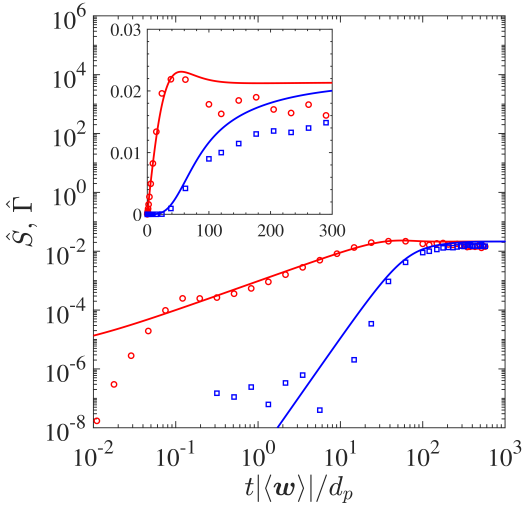

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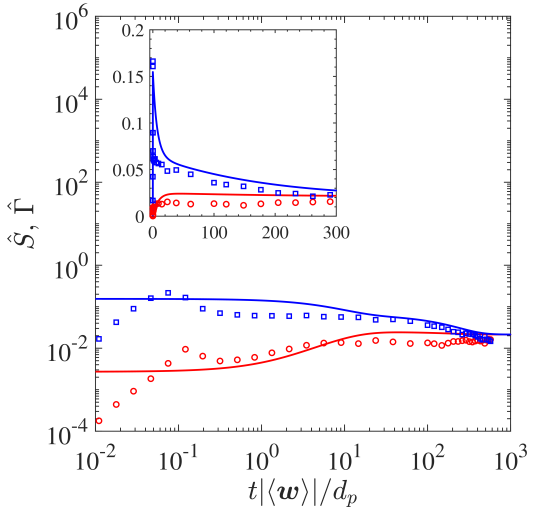

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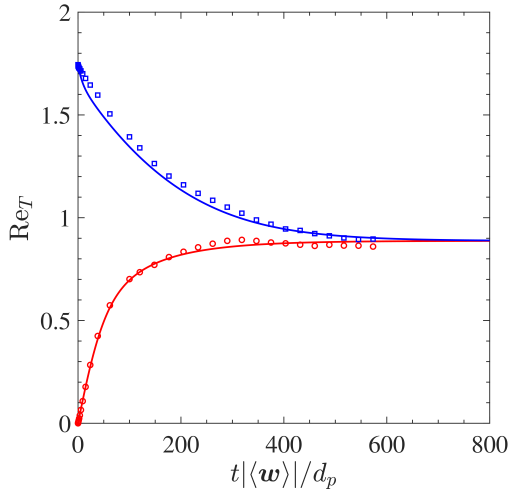

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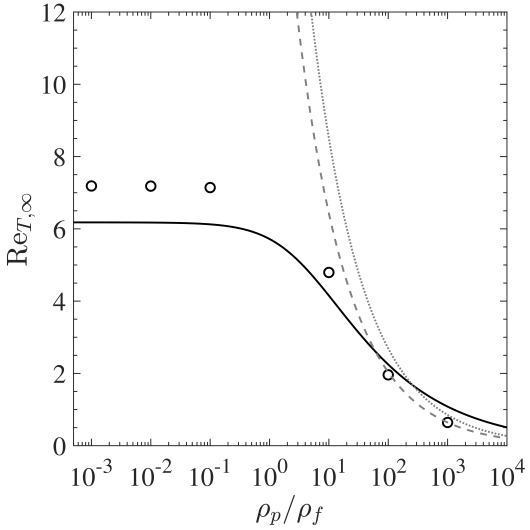

1. Capture $S(t)$, $\Gamma(t)$, $T(t)$ in HHS & HCS
2. Well behaved extrapolation: $\rho_p/\rho_f \ll 100$
3. Predicts T_∞ saturation for $\rho_p/\rho_f \ll 1$
4. Potetial general theory from gas-solid to bubbly flows

Questions?