Stochastic modeling of drag forces in Euler-Lagrange simulations of particle-laden flows

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Drag force(s) acting on a collection of particles

Suspensions exhibit significant drag variation

Shallcross et al. (IP)  
Akiki et al. (2016)

Fullmer et al. (2016)

1 Tenneti et al. (2016)  
2 Akiki et al. (2017)  
3 Esteghamatian et al. (2018)
Drag force(s) acting on a collection of particles

Suspensions exhibit significant drag variation

Existing drag laws fail to capture higher-order (HO) statistics $^{1-3}$

- Particle velocity variance
- Particle dispersion

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Why?

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$^2$ Akiki et al. (2017)
$^3$ Esteghamatian et al. (2018)
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Existing drag laws fail to capture higher-order (HO) statistics $^{1-3}$
- Particle velocity variance
- Particle dispersion

Why?
They coarse-grain the microstructure...

$U$

References:
1. Tenneti et al. (2016)
2. Akiki et al. (2017)
3. Esteghamatian et al. (2018)
Drag force(s) acting on a collection of particles

Suspensions exhibit significant drag variation. Existing drag laws fail to capture higher-order (HO) statistics, specifically:

- Particle velocity variance
- Particle dispersion

Why? They coarse-grain the microstructure...

Shallcross et al. (2014)  
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Esteghamatian et al. (2018)

Hydrodynamic forces:

\[ m_p^{(i)} \frac{dU_p^{(i)}}{dt} = V_p^{(i)} \nabla \cdot \vec{\tau} \left[ X_p^{(i)} \right] + \int \tau' \cdot n \, dS \]

**Isolated sphere (Maxey-Riley)**

\[ 3\pi \mu_f d_p f_{iso} \left( u_f - U_p^{(i)} \right) + \frac{\rho_f V_p^{(i)}}{2} \frac{d}{dt} \left( u_f - U_p^{(i)} \right) + \frac{3}{2} \sqrt{\pi \rho_f \mu_f} d_p^2 \int_0^t \left[ \frac{d}{d\tau} \left( u_f - U_p^{(i)} \right) \right] \frac{(t - \tau)^{1/2}}{d\tau} \, d\tau \]

Quasi-steady drag  
Added mass  
Basset history

We seek a stochastic framework that incorporates these effects in the drag statistics of a suspension.

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1 Tenneti et al. (2016)  
2 Akiki et al. (2017)  
3 Esteghamatian et al. (2018)
Talk forecast

Emphasize stochastic EL framework

- Hydrodynamic forces
- HO particle moments
- Statistical approach

1. **Stochastic hierarchy**
   - Langevin Eqs.

2. **Stochastic EL solver**
   - Improved predictions

3. **Closure for EE solvers**
   - Hydrodynamic sources

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1. Lattanzi et al. (2020)
2. Lattanzi et al. (2021)
3. Lattanzi et al. (IP)
Neighbor-induced drag statistics

PR-DNS studies show Gaussian PDFs

1, 2, 3

Huang et al. (2017)  Balachandar (2020)

Esteghamatian et al. (2018)

1 Akiki et al. (2017)
2 Seyed-Ahmadi et al. (2020)
3 Lattanzi et al. (2020)
Neighbor-induced drag statistics

PR-DNS studies show Gaussian PDFs \(^1\)–\(^3\)

Expand unresolved drag about the mean

\[
\int \tau' \cdot n \, dS = \langle F_d \rangle + F_d''(i)
\]

**Figure:**

1. Akiki *et al.* (2017)
2. Seyed-Ahmadi *et al.* (2020)
3. Lattanzi *et al.* (2020)
Neighbor-induced drag statistics

PR-DNS studies show Gaussian PDFs \(^1\)–\(^3\)

Expand unresolved drag about the mean
\[
\int \tau' \cdot n \, dS = \langle F_d \rangle + F''(i)
\]

- Specify statistics via \(F''(i)\)
  - Deterministic PIEP \(^1\)
  - Stochastic Langevin \(^3\)

\(^1\) Akiki et al. (2017)
\(^2\) Seyed-Ahmadi et al. (2020)
\(^3\) Lattanzi et al. (2020)

Esteghamatian et al. (2018)

Huang et al. (2017)

Balachandar (2020)
Neighbor-induced drag statistics

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Expand unresolved drag about the mean
\[ \int \tau' \cdot n \, dS = \langle F_d \rangle + F''(i) \]

- Specify statistics via \( F''(i) \)
  - Deterministic PIEP
  - Stochastic Langevin

Choose \( \langle F_d \rangle \) correlation

- \( \langle F_d \rangle = f (Re_p, \phi) \)
- \( \langle F_d \rangle = f (Re_p, \phi, \rho_p/\rho_f) \)

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1 Akiki et al. (2017)
2 Seyed-Ahmadi et al. (2020)
3 Lattanzi et al. (2020)
Consequence of force Langevin equation

\[
dF_d^{'''}(i) = -\frac{1}{\tau_F} F_d^{'''}(i) \, dt + \frac{\sqrt{2} \sigma_F}{\sqrt{\tau_F}} \, dW_t
\]

Evolution of particle-phase moments

- Gaussian drag fluctuations
- Exponential ACF

Lattanzi et al. (2020)
Consequence of force Langevin equation

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Evolution of particle-phase moments

- Gaussian drag fluctuations
- Exponential ACF
- Source of velocity variance
- Source of dispersion

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Evolution of particle-phase moments

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Higher-order drag statistics consistent with PR-DNS
Towards a stochastic drag framework

Equation of motion:

\[ m_p^{(i)} \frac{dU_p^{(i)}}{dt} = \sum_{j=1}^{N} F_{\text{col}}^{(ij)} + V_p^{(i)} \nabla \cdot \bar{\tau} \left[ X_p^{(i)} \right] + \int \tau' \cdot n \; dS \]
Towards a stochastic drag framework

Equation of motion:

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\[ \int \tau' \cdot n \, dS = \left\langle F_d^{(i)} \right\rangle + F_d''^{(i)} \]
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\[ \int \tau' \cdot n \, dS = \left\langle F_d^{(i)} \right\rangle + F_d''^{(i)} \]

\[ \left\langle F_d^{(i)} \right\rangle = f(Re_p, \phi) \]

Tenneti et al. (2011)
Towards a stochastic drag framework

Equation of motion:

\[ m_p^{(i)} \frac{dU_p^{(i)}}{dt} = \sum_{j=1}^{N} F_{\text{col}}^{(ij)} + V_p^{(i)} \nabla \cdot \bar{T} \left[ X_p^{(i)} \right] + \int \tau' \cdot n \, dS \]

\[ \int \tau' \cdot n \, dS = \left\langle F_d^{(i)} \right\rangle + F''_d^{(i)} \]

\[ \left\langle F_d^{(i)} \right\rangle = f \left( \text{Re}_p, \phi \right) \]

Tenneti et al. (2011)
Towards a stochastic drag framework

Equation of motion:

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\]

\[
\int \tau' \cdot n \, dS = \left< F_d^{(i)} \right> + F_d^{''(i)}
\]

\[
\left< F_d^{(i)} \right> = f(Re_p, \phi)
\]

\[
dF_d^{''(i)} = -\frac{1}{\tau_F} F_d^{''(i)} \, dt + \frac{\sqrt{2} \sigma_F}{\sqrt{\tau_F}} \, dW_t
\]

Tenneti et al. (2011)

\[
\tau_F \approx \tau_{col} = \frac{d_p}{24 \phi \chi} \sqrt{\frac{\pi}{T}}
\]

Chapman & Cowling (1970)
Towards a stochastic drag framework

Equation of motion:

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m_p^{(i)} \frac{dU_p^{(i)}}{dt} = \sum_{j=1}^{N} F_{\text{col}}^{(ij)} + V_p^{(i)} \nabla \cdot \bar{\tau} \left[ X_p^{(i)} \right] + \int \tau' \cdot n \, dS
\]

\[
\int \tau' \cdot n \, dS = \langle F_d^{(i)} \rangle + F_d''^{(i)}
\]

\[
\langle F_d^{(i)} \rangle = f (\text{Re}_p, \phi)
\]

\[
dF_d''^{(i)} = -\frac{1}{\tau_F} F_d''^{(i)} \, dt + \frac{\sqrt{2} \sigma_F}{\sqrt{\tau_F}} \, dW_t
\]

Tenneti et al. (2011)

\[
\tau_F \approx \tau_{\text{col}} = \frac{d_p}{24 \phi \chi} \sqrt{\frac{\pi}{T}}
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Chapman & Cowling (1970)

Need model for \( \sigma_F \)
A correlation for drag variance

PR-DNS of fixed assemblies

- PUReIBM

Tavanashad & Subramaniam (2020)
A correlation for drag variance

PR-DNS of fixed assemblies

- PUReIBM

Tavanashad & Subramaniam (2020)

\[
f^\sigma_F = 6.52\phi - 22.56\phi^2 + 49.90\phi^3
\]
A correlation for drag variance

PR-DNS of fixed assemblies

- PUReIBM

\[
\sigma_F \equiv \frac{m_p^{\phi} F_{\text{single}}}{m_p} = f_{\phi}^\sigma f_{\text{iso}} \frac{(1 - \phi) \left\| \mathbf{u_f} \left[ X_p^{(i)} \right] - U_p^{(i)} \right\|}{\tau_p}
\]

\[
f_{\phi}^\sigma = 6.52 \phi - 22.56 \phi^2 + 49.90 \phi^3
\]

\[
f_{\text{iso}} = (1 + 0.15 \text{Re}_p^{0.687})
\]

Tavanashad & Subramaniam (2020)
EL framework

NGA low mach solver

- Volume filtering, $2^{\text{nd}}$ order scheme

\[
\frac{\partial}{\partial t} ((1 - \phi) \rho_f) + \nabla \cdot ((1 - \phi) \rho_f u_f) = 0
\]

\[
\frac{\partial}{\partial t} ((1 - \phi) \rho_f u_f) + \nabla \cdot ((1 - \phi) \rho_f u_f \otimes u_f) = \nabla \cdot \bar{\tau} + (1 - \phi) \rho_f g - \mathcal{F}_{\text{inter}} + \mathcal{F}_{\text{mfr}}
\]

Lagrangian particle tracking

- Soft-sphere (multiple, enduring contacts)

\[
\frac{dX_p^{(i)}}{dt} = U_p^{(i)}
\]

\[
m_p^{(i)} \frac{dU_p^{(i)}}{dt} = \sum_{j=1}^{N} F_{\text{col}}^{(ij)} + F_{\text{inter}}^{(i)} + m_p^{(i)} g
\]

\[
F_{\text{inter}}^{(i)} = V_p^{(i)} \nabla \cdot \bar{\tau} \left[ X_p^{(i)} \right] + \left\langle F_d^{(i)} \right\rangle + F_d''^{(i)}
\]
Homogeneous fluidization of elastic particles

**Triply periodic box** $^{1-2}$

- Force flow rate $\langle |\mathbf{W}| \rangle$
- Gravity opposes flow $\mathbf{g}$
- $\rho_p/\rho_f$, $\text{Re}_m$, $\phi$

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Homogeneous fluidization of elastic particles

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- Force flow rate $\langle |\mathbf{W}| \rangle$
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- $\rho_p/\rho_f$, $\text{Re}_m$, $\phi$

**Fluidized homogeneous heating system (FHHS)**

- Velocity IC $\delta(u)$

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Homogeneous fluidization of elastic particles

Triply periodic box \(^{1-2}\)
- Force flow rate \(< |\mathbf{W}| >\)
- Gravity opposes flow \(\mathbf{g}\)
- \(\rho_p/\rho_f, \text{Re}_m, \phi\)

Fluidized homogeneous heating system (FHHS)
- Velocity IC \(\delta(\mathbf{u})\)

Fluidized homogeneous cooling system (FHCS)
- Over-prescribed variance \(\mathcal{N}[0, \sigma_v, 0]\)

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1. Tenneti et al. (2016)
2. Tavanashad et al. (2020)
Fixed conditions: $\frac{\rho_p}{\rho_f} = 100$, $\phi = 0.1$

- Stochastic EL (---)  Standard EL (---)  PR-DNS (○)

Stochastic FL captures growth and steady velocity variance
**Fixed conditions:** $Re_m = 20, \ \rho_p/\rho_f = 100, \ \phi = 0.1$

- Stochastic EL (---)  
  - Standard EL (---)  
  - PR-DNS (○)

Stochastic FL captures decay and steady velocity variance
FHHS: $\phi, \rho_p/\rho_f$ sweep

Fixed conditions: $Re_m = 20$

- Stochastic EL (—) PR-DNS (○)

Stochastic FL captures dynamics over wide range
Talk forecast: extensions

Emphasize stochastic EE framework

- Hydrodynamic forces
- HO particle moments
- Statistical approach

1. Stochastic hierarchy
   - Langevin Eqs.

2. Stochastic EL solver
   - Improved predictions

3. Closure for EE solvers
   - Hydrodynamic sources

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1. Lattanzi et al. (2020)
2. Lattanzi et al. (2021)
3. Lattanzi et al. (IP)
Extension to Euler-Euler frameworks

Homogeneous, smooth, elastic spheres

\[ \frac{dT}{dt} \equiv S - \Gamma = \frac{2}{3} \langle A_i' V_i' \rangle \quad T = \frac{1}{3} \text{Tr} \left( \langle V'_p \otimes V'_p \rangle \right) \]
Extension to Euler-Euler frameworks

Homogeneous, smooth, elastic spheres

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Tenneti et al. (2016)
Extension to Euler-Euler frameworks

Homogeneous, smooth, elastic spheres

$$\frac{dT}{dt} \equiv S - \Gamma = \frac{2}{3} \langle A_i' V_i' \rangle$$

$$T = \frac{1}{3} \text{Tr} \left( \langle V'_p \otimes V'_p \rangle \right)$$

Require solution to Fokker-Planck

$$\frac{\partial P(v', a''; t)}{\partial t} + \frac{\partial}{\partial v'} \left[ (a'' - \frac{1}{\tau_d} v') P \right] - \frac{1}{\tau_{a''}} \frac{\partial}{\partial a''} (a'' P) = \sigma_{a''}^2 \frac{\partial^2 P}{\tau_{a''} \partial a''^2}.$$
Extension to Euler-Euler frameworks

Homogeneous, smooth, elastic spheres

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Tenneti et al. (2016)
Extension to Euler-Euler frameworks

HHS $S \& \Gamma$

HCS $S \& \Gamma$

Density ratio $\rho_p / \rho_f \ll 1$

Well behaved extrapolation:

Predicts $T_\infty$ saturation for $\rho_p / \rho_f \ll 1$

Potential general theory from gas-solid to bubbly flows
Extension to Euler-Euler frameworks

HHS $S$ & $\Gamma$

HCS $S$ & $\Gamma$

$T(t)$

Density ratio $\rho_p/\rho_f \ll 100$

Predicts $T_\infty$ saturation for $\rho_p/\rho_f \ll 1$
Extension to Euler-Euler frameworks

**HHS $S$ & $\Gamma$**

**HCS $S$ & $\Gamma$**

$T(t)$

**Density ratio**

Well behaved extrapolation: $\rho_p/\rho_f \ll 100$

Predicts $T_\infty$ saturation for $\rho_p/\rho_f \ll 1$

Potential general theory from gas-solid to bubbly flows
Extension to Euler-Euler frameworks

**HHS $S$ & $\Gamma$**

**HCS $S$ & $\Gamma$**

**$T(t)$**

**Density ratio**

1. Capture $S(t)$, $\Gamma(t)$, $T(t)$ in HHS & HCS
2. Well behaved extrapolation: $\rho_p/\rho_f \ll 100$
3. Predicts $T_\infty$ saturation for $\rho_p/\rho_f \ll 1$
4. Potential general theory from gas-solid to bubbly flows
Questions?