An Easily Implementable General Self-induced Perturbation Correction Model for a Finite-sized Particle in Two-way Coupled Euler-Lagrange Simulations

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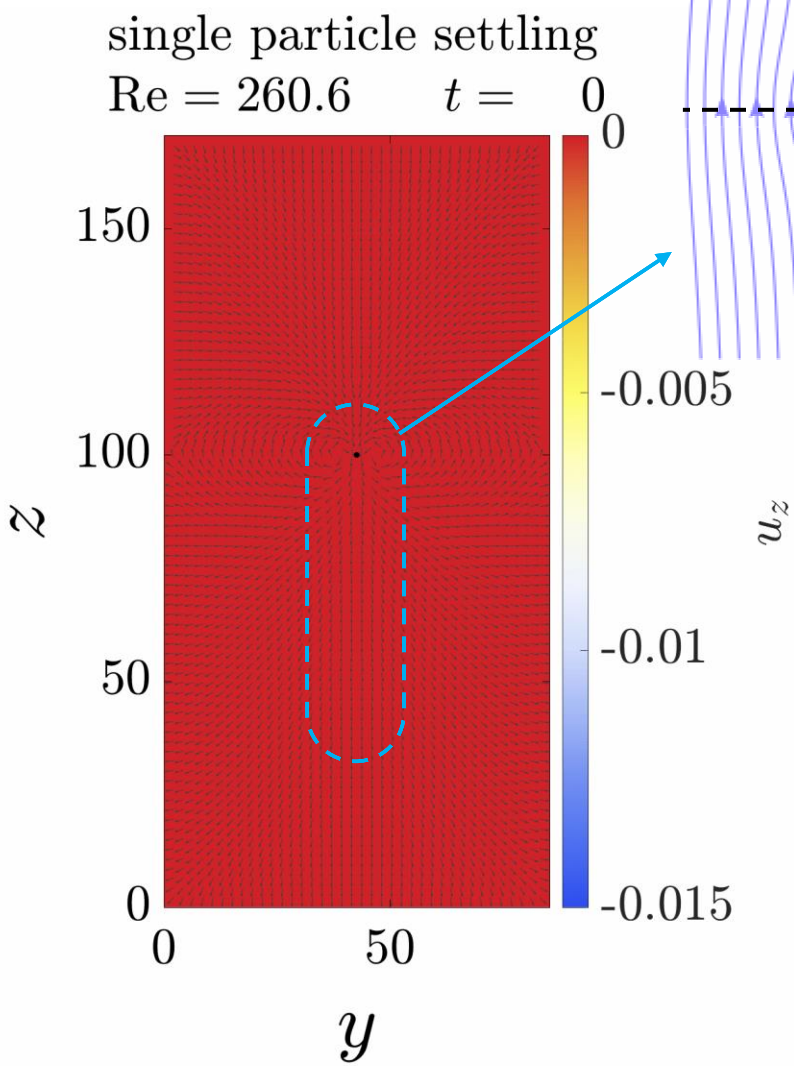
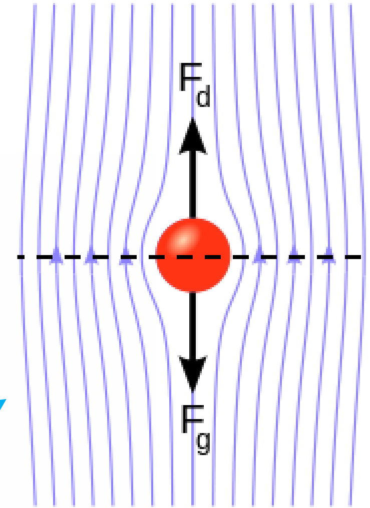
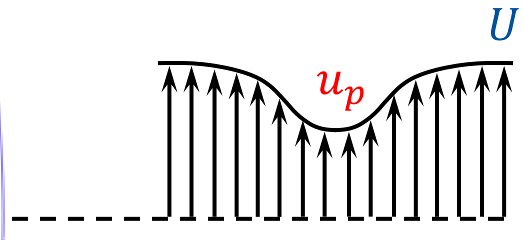
Motivation

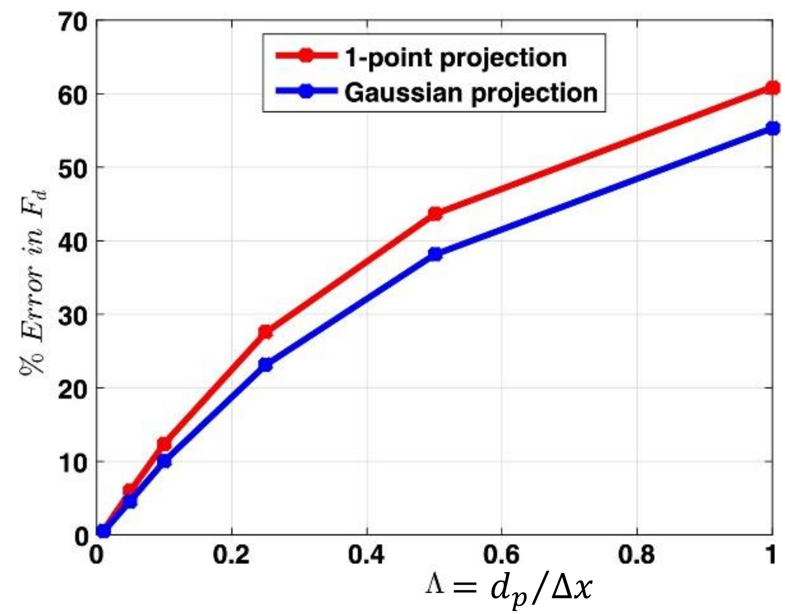






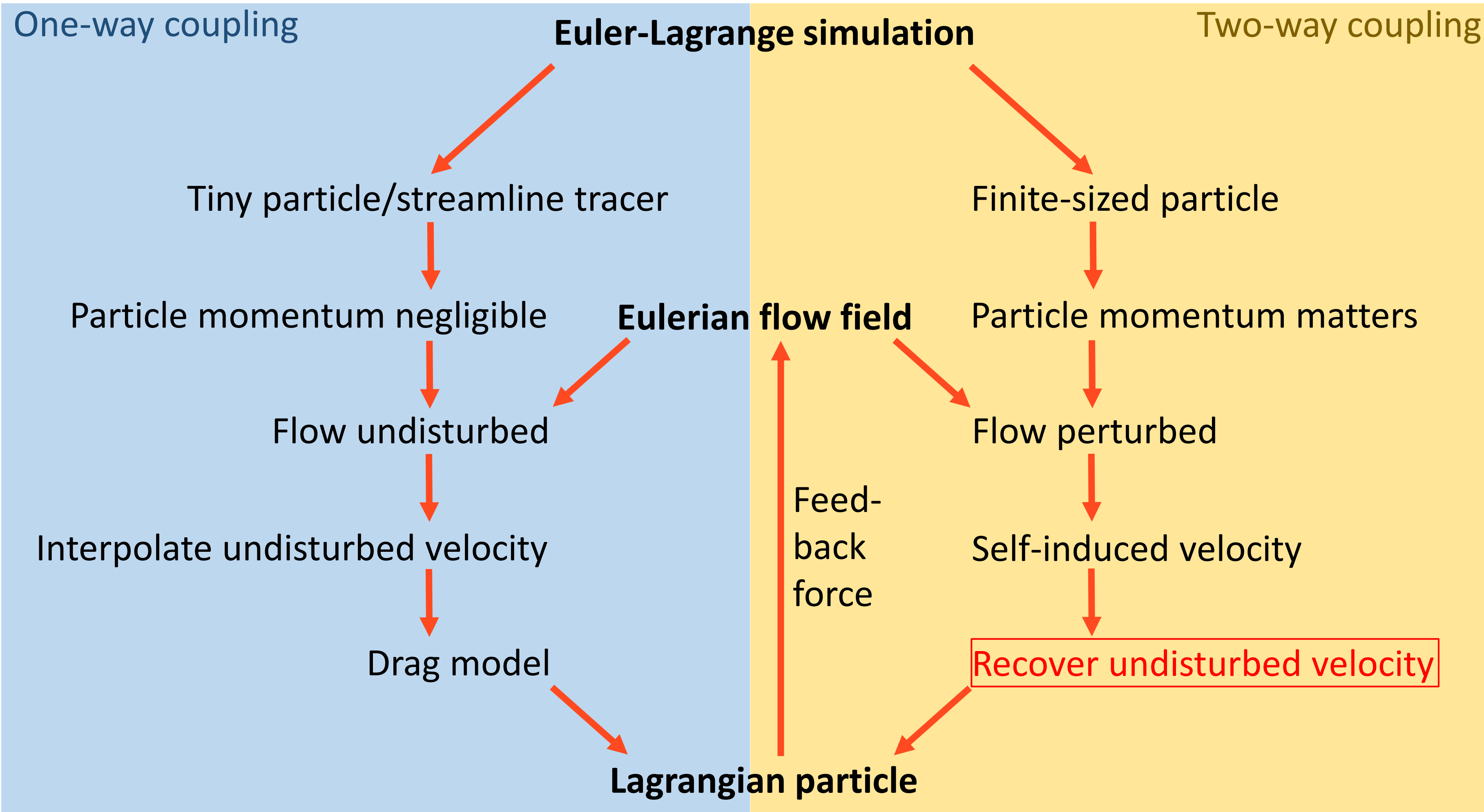
Stokes' drag:

$$F_d = \begin{cases} 3\pi\mu d_p(U - v_p) & \checkmark \text{ By theory} \\ 3\pi\mu d_p(u_p - v_p) & \times \text{ In simulation} \end{cases}$$



J. A. K. Horwitz and A. Mani. Journal of Computational Physics, 2016.

Hierarchy



One-way coupling
Two-way coupling
Euler-Lagrange simulation
Tiny particle/streamline tracer
Finite-sized particle
Particle momentum negligible
Eulerian flow field
Particle momentum matters
Flow undisturbed
Flow perturbed
Feed-
back
force
Interpolate undisturbed velocity
Self-induced velocity
Drag model
Recover undisturbed velocity
Lagrangian particle

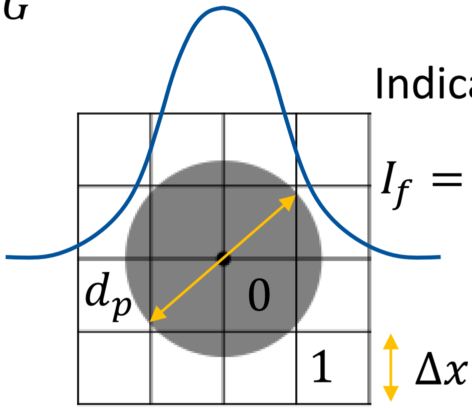
Our plan

- Simple model
- Drag not in same direction of flow
- Near wall
- Compatible for common projection filters
- Compatible for common difference schemes

Oseen's equation:

$$\nabla \cdot (\phi_{c1} \boldsymbol{u}'_{1c}) = 0$$
$$\rho \boldsymbol{U} \cdot (\phi_{c1} \boldsymbol{u}'_{1c}) = -\nabla(\phi_{c1} p'_{1c}) + \mu \nabla^2 (\phi_{c1} \boldsymbol{u}'_{1c}) - \boldsymbol{F}G$$

$$G(\boldsymbol{r}) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^3} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$



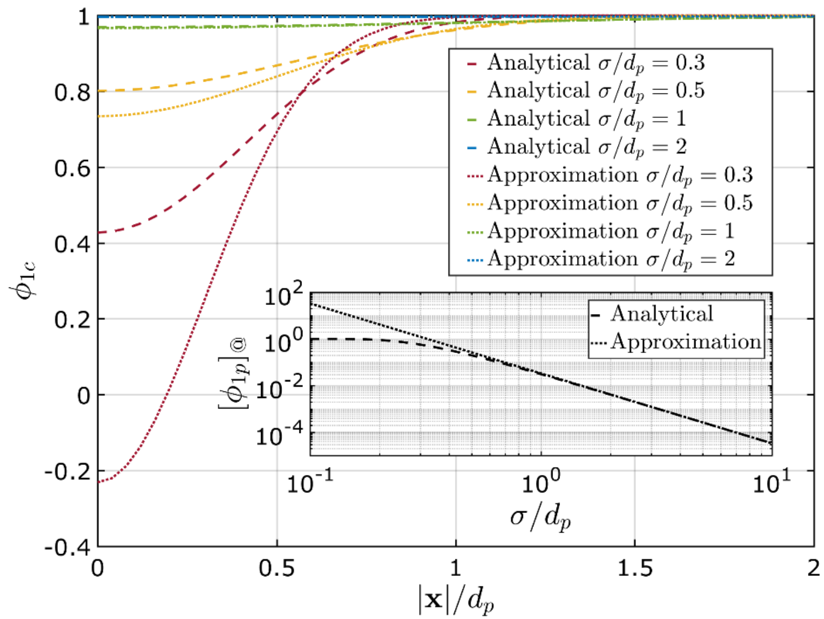
Indicator function

$$I_f = \begin{cases} 0 \text{ Particle} \\ 1 \text{ Fluid} \end{cases}$$

Solution:

$$\underbrace{\boldsymbol{U}(\boldsymbol{x}, t)}_{\text{Undisturbed velocity}} = \underbrace{\phi_{c1} \boldsymbol{u}_c(\boldsymbol{x}, t)}_{\text{EL particle Center velocity}} - \underbrace{\phi_{c1} \boldsymbol{u}'_{1c}(\boldsymbol{x}, t)}_{\text{Self-induced perturbation velocity}}$$

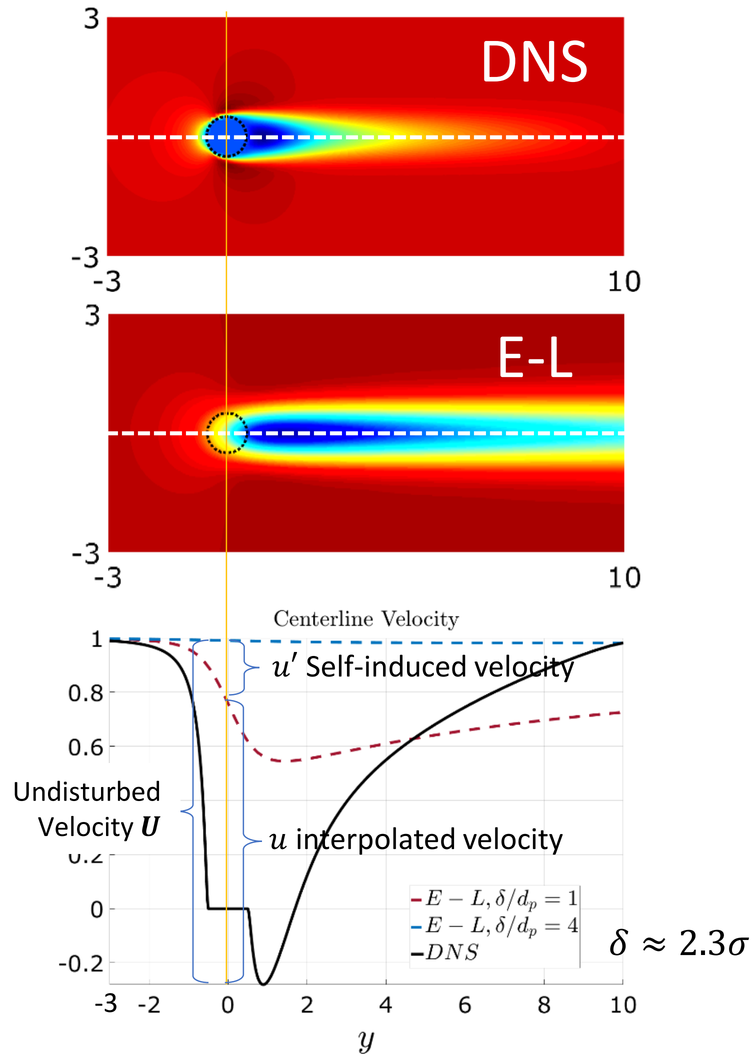
Filter effect



$$\phi_{1c} = \begin{cases} 1 - \mathrm{erf}\left(\frac{d_p}{2\sqrt{2}\sigma}\right) + \frac{d_p}{\sqrt{2\pi}\sigma}\exp\left(-\frac{d_p^2}{8\sigma^2}\right) & \text{Analytical} \\ 1 - \frac{\pi d_p^3}{6}\frac{1}{(\sqrt{2\pi}\sigma)^3}\exp\left(-\frac{0^2}{2\sigma^2}\right) & \text{Approximation} \end{cases}$$

- Recommend using the analytical form
- σ/d_p ↘ ⟶ fluid volume fraction ↗
- σ/d_p ↘ ⟶ self-induced velocity ↗

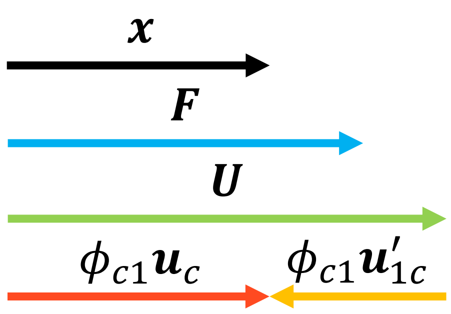
$U = 1, d_p = 1, Re_p = 170$

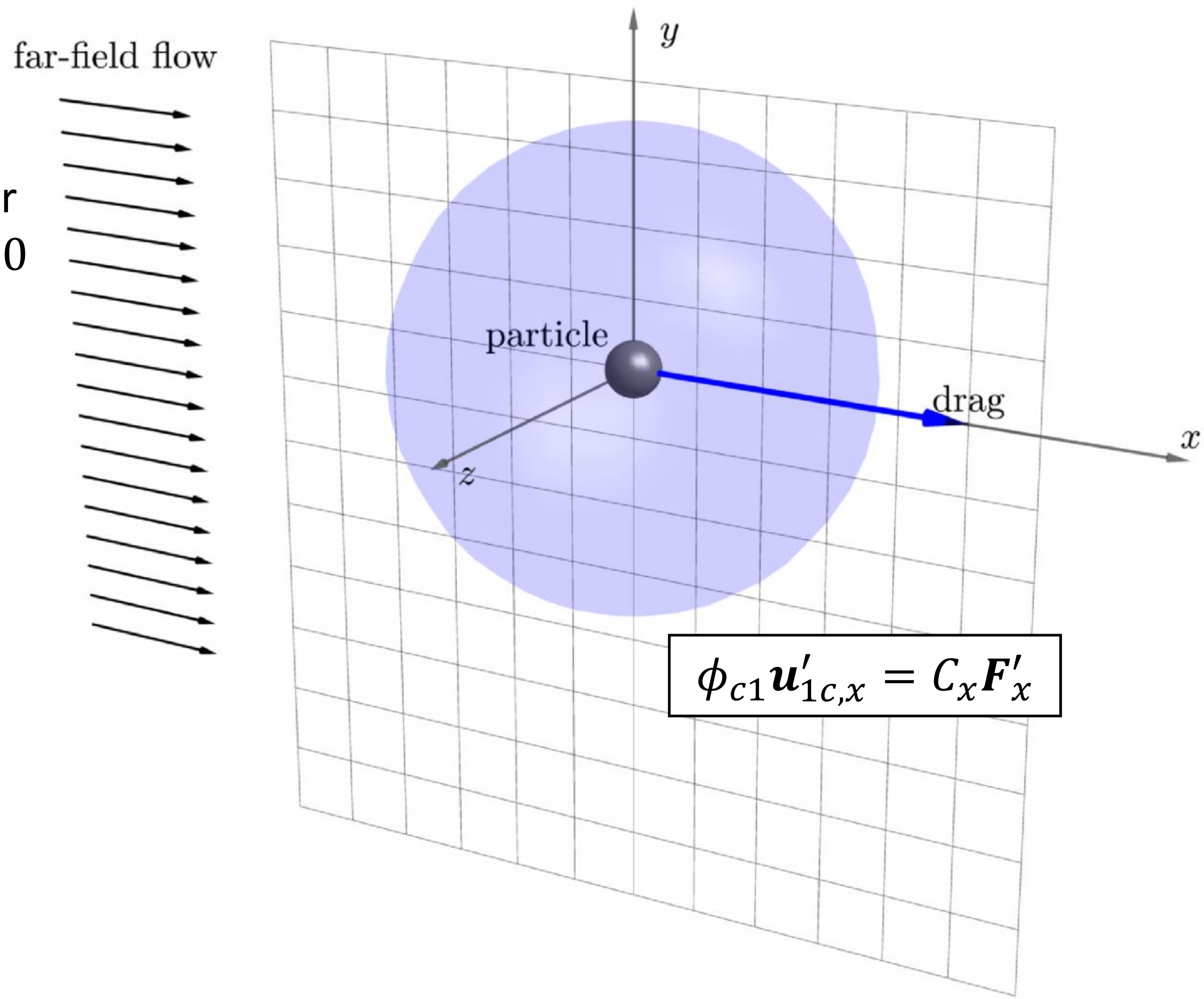


$\delta \approx 2.3\sigma$

Scenario 1: x-dir scalar

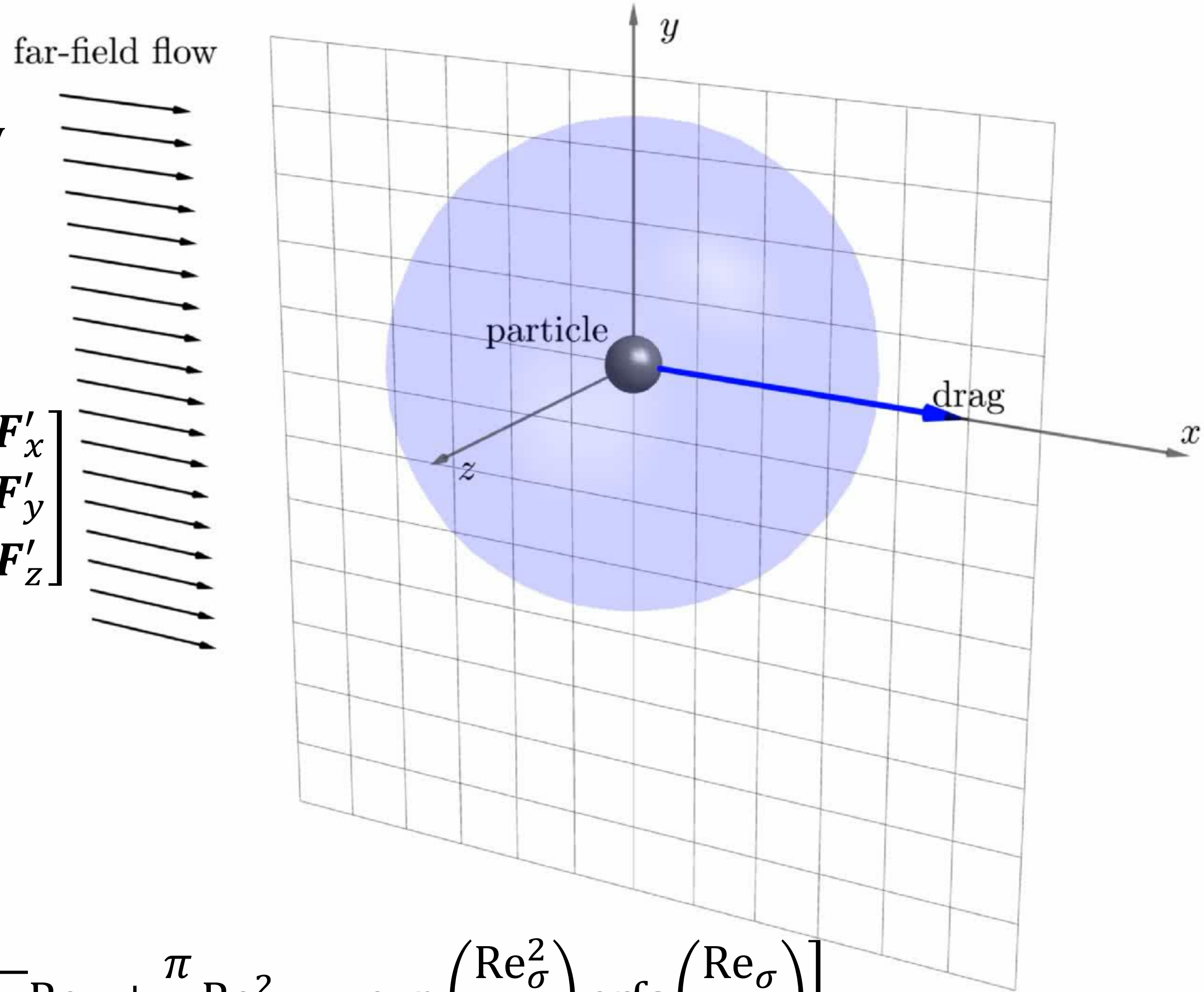
- Drag is parallel to flow
- self-induced velocity in x-dir
- No y,z perturbation @ $\boldsymbol{x} = 0$





Scenario 2: 3D vectorial

- Drag is not parallel to flow
- Due to not only $\boldsymbol{F}'_{qs}$
- Diagonal relation



$$\begin{bmatrix} \phi_{c1}\boldsymbol{u}'_{1c,x} \\ \phi_{c1}\boldsymbol{u}'_{1c,y} \\ \phi_{c1}\boldsymbol{u}'_{1c,z} \end{bmatrix} = \begin{bmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_z \end{bmatrix} \begin{bmatrix} \boldsymbol{F}'_x \\ \boldsymbol{F}'_y \\ \boldsymbol{F}'_z \end{bmatrix}$$

$$\mathrm{Re}_\sigma = \frac{|\overline{\boldsymbol{U}}^s(\boldsymbol{x}=\boldsymbol{0},t) - \boldsymbol{V}|\sigma}{\nu}$$

$$M_x(\mathrm{Re}_\sigma) = -\frac{1}{2\pi^2\mathrm{Re}_\sigma^2}\left[\pi - \sqrt{2\pi}\mathrm{Re}_\sigma + \frac{\pi}{2}\mathrm{Re}_\sigma^2 - \pi\exp\left(\frac{\mathrm{Re}_\sigma^2}{2}\right)\mathrm{erfc}\left(\frac{\mathrm{Re}_\sigma}{\sqrt{2}}\right)\right]$$

Scenario 3: near-wall

- Free-slip wall nearby
- Projection point in x-y plane

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- Modelled by a mirror particle
- Symmetric by the wall

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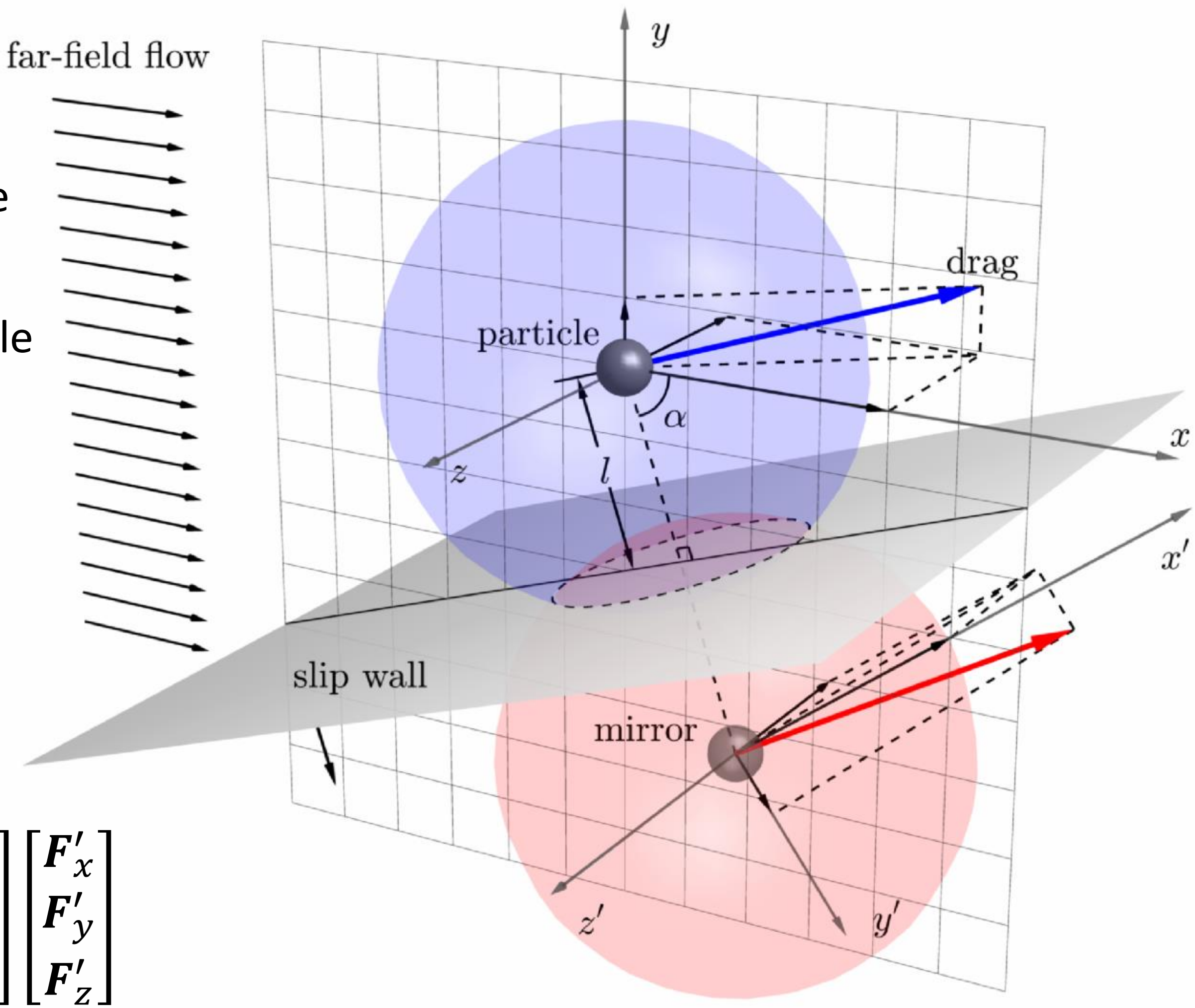
- Drag symmetric by the wall
- Not only diagonal relation



$$\begin{bmatrix} \phi_{c1}\boldsymbol{u}'_{1c,x} \\ \phi_{c1}\boldsymbol{u}'_{1c,y} \\ \phi_{c1}\boldsymbol{u}'_{1c,z} \end{bmatrix} = \begin{bmatrix} M_{xx} & M_{xy} & 0 \\ M_{yx} & M_{yy} & 0 \\ 0 & 0 & M_{zz} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}'_x \\ \boldsymbol{F}'_y \\ \boldsymbol{F}'_z \end{bmatrix}$$

Correction model

Start

Interpolated relative flow velocity $\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}$

standard drag

$$\boldsymbol{F}'_{qs} = 3\pi\mu d_p[\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}]\Phi$$

Coordinates rotation $[\mathbf{M}][\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}]$

Compute $\mathrm{Re}_\sigma, \mathrm{Re}_p, \Phi, [\mathbf{B}], [\mathbf{C}]$ using $[\mathbf{M}][\overline{\boldsymbol{U}}^s - \boldsymbol{V}]$

$$\overline{\boldsymbol{U}}^s - \boldsymbol{V} = [\mathbf{M}]^{\boldsymbol{T}}\left\{[\boldsymbol{I}] - \frac{d_p}{\sigma}\frac{\Phi}{\sqrt{2\pi}}[\mathbf{B}]([\boldsymbol{I}] + [\mathbf{C}])\right\}^{-\mathbf{1}}[\mathbf{M}]\left(\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}\right)$$

Renew $\overline{\boldsymbol{U}}^s - \boldsymbol{V}$ and Iterate

Converge

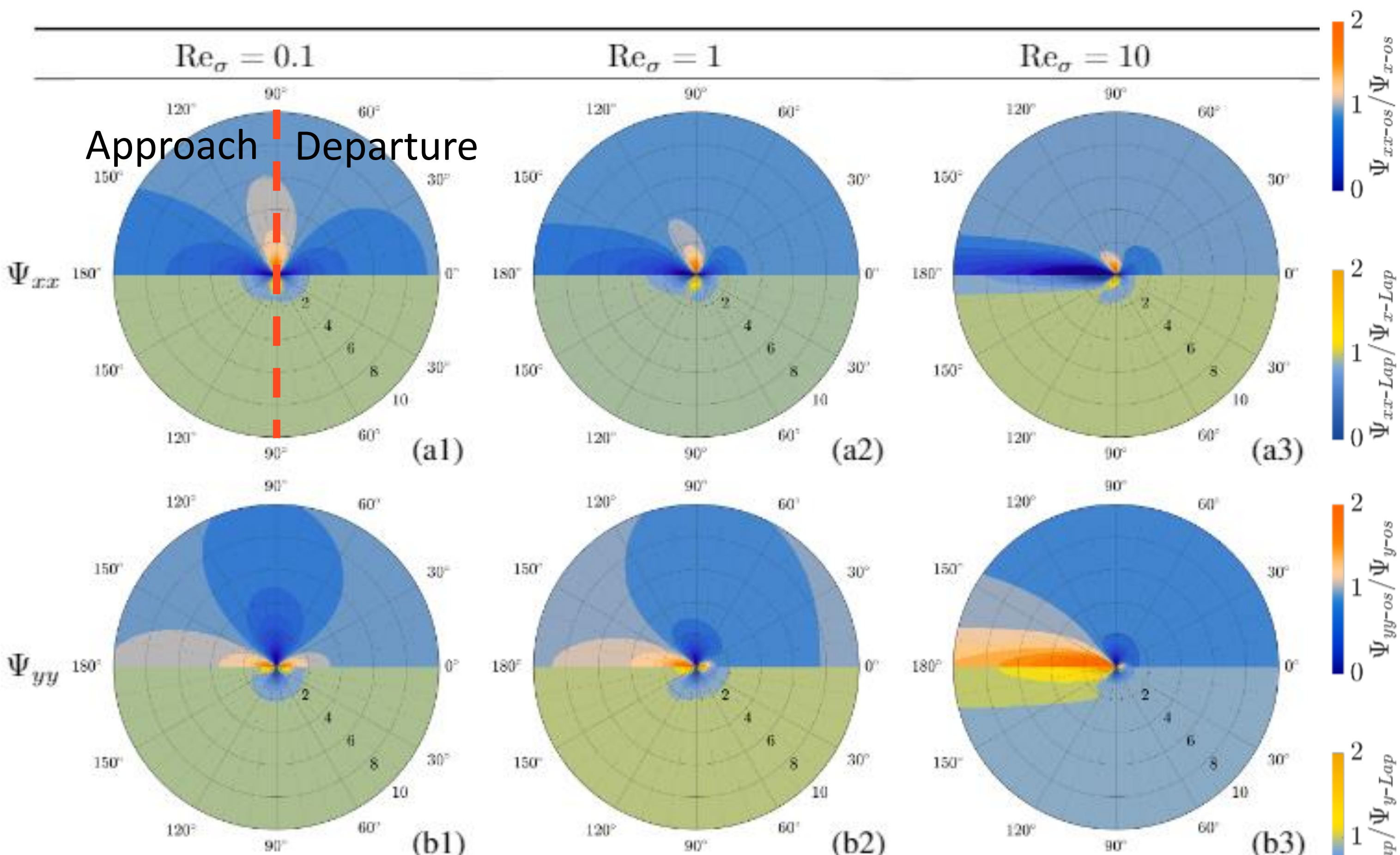
True drag $\boldsymbol{F}'_{qs} = 3\pi\mu d_p[\overline{\boldsymbol{U}}^s - \boldsymbol{V}]\Phi$

$$[\mathbf{B}] = \begin{bmatrix} \Psi_{xx-os} & \Psi_{xy-os} & 0 \\ \Psi_{yx-os} & \Psi_{yy-os} & 0 \\ 0 & 0 & \Psi_{zz-os} \end{bmatrix} - \frac{d_p^2}{24\sigma^2}\begin{bmatrix} \Psi_{xx-Lap} & \Psi_{xy-Lap} & 0 \\ \Psi_{yx-Lap} & \Psi_{yy-Lap} & 0 \\ 0 & 0 & \Psi_{zz-Lap} \end{bmatrix}$$

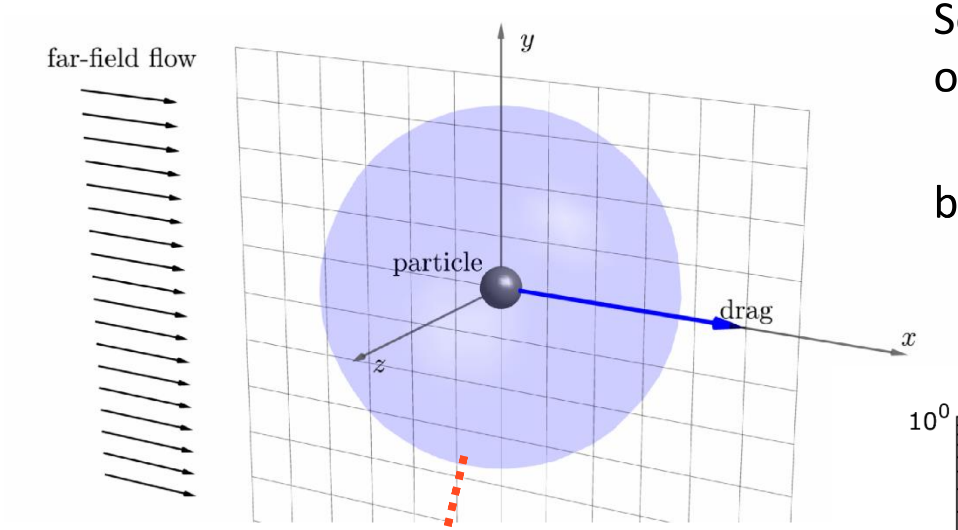
$$[\mathbf{C}] = [\mathbf{M}]\left[\frac{\boldsymbol{F}'_{ot}}{\boldsymbol{F}'_{qs}}\right]\left[\frac{\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}}{\overline{\boldsymbol{U}}^s - \boldsymbol{V}}\right][\mathbf{M}]^{\boldsymbol{T}}$$

Ψ components

Obtained totally 10 components of $\Psi\left(\mathrm{Re}_\sigma, \frac{l}{\sigma}, \alpha\right)$ with explicit asymptotic functions for low Re_σ



Different filters



$$G^1(r) = \frac{1}{(2\pi)^{3/2}\sigma^3}\exp\left(-\frac{r^2}{2\sigma^2}\right)$$

$$G^2(r) = \frac{1}{\pi\Gamma\left(\frac{3}{4}\right)(\ln 16)^{3/4}\sigma^3}\exp\left[-\frac{r^4}{(\ln 16)\sigma^4}\right]$$

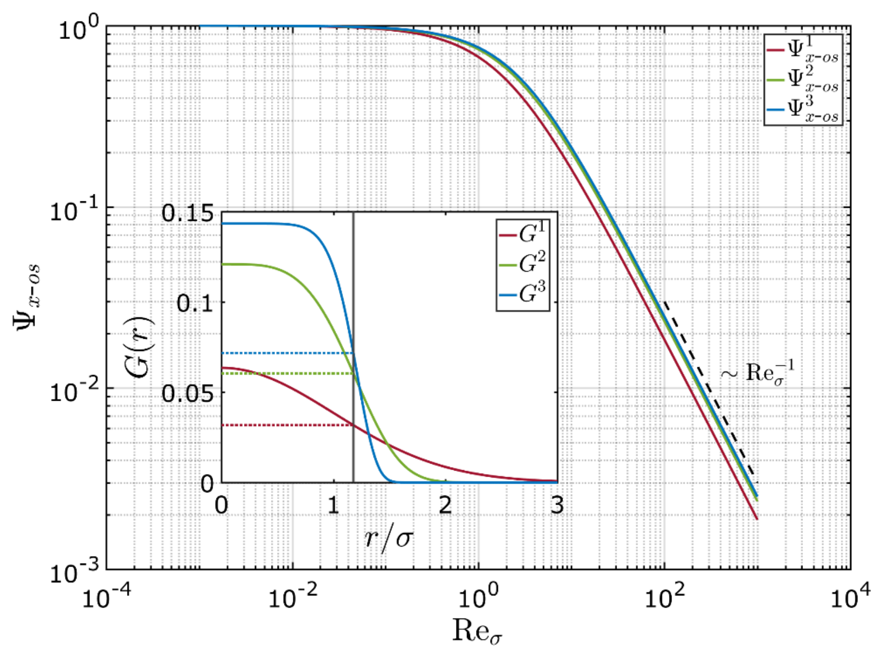
$$G^3(r) = \frac{2^{5/8}}{\pi\Gamma\left(\frac{3}{8}\right)(\ln 4)^{9/8}\sigma^3}\exp\left[-\frac{r^8}{2(\ln 4)^3\sigma^8}\right]$$

Scalar case for example, different choice of filter does not change general solution

$$\overline{\phi_{c1}\boldsymbol{u}_c'}^s = C\mathrm{Re}_\sigma \boldsymbol{F}_x' \Psi_{x-os}$$

but slightly changes the specific values of

$$C\,,\Psi_{x-os}$$



Differentiation schemes

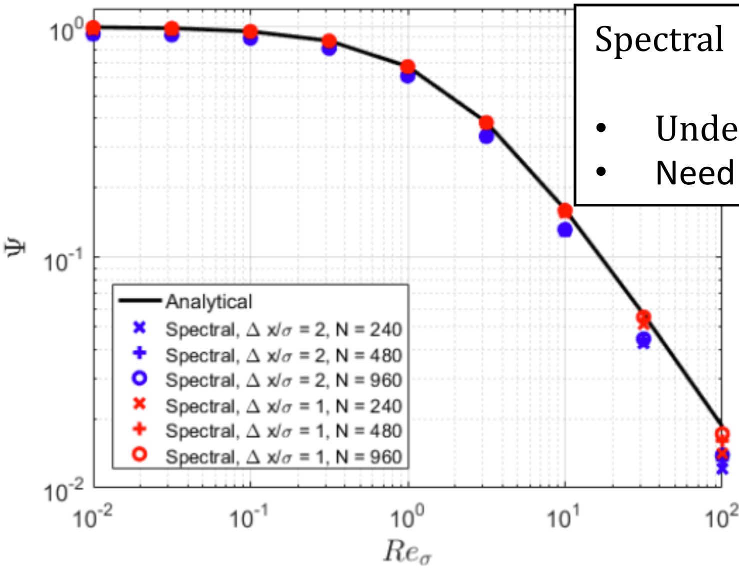
Self-induced velocity derived by Fourier Transform $\widehat{\phi_{c1}\boldsymbol{u}'_c} = -\frac{\boldsymbol{F}'_x \mathrm{Re}_\sigma \left(1-\iota\frac{\mathrm{Re}_\sigma \tilde{k}_x}{\left|\tilde{\tilde{k}}\right|^2}\right)}{\left|\tilde{\tilde{k}}\right|^2\left(1+\frac{\mathrm{Re}_\sigma^2 \tilde{k}^2}{\left|\tilde{\tilde{k}}\right|^4}\right)}\left(1-\frac{\tilde{k}_x^2}{\tilde{k}^2}\right)\hat{G}$

Wave number in spectral method

$$\tilde{k} = k \text{ and } \tilde{\tilde{k}}^2 = k^2 \text{ for } k = 0, \pm k_{min}, \pm 2k_{min}, \cdots, \pm\frac{N}{2}k_{min}$$

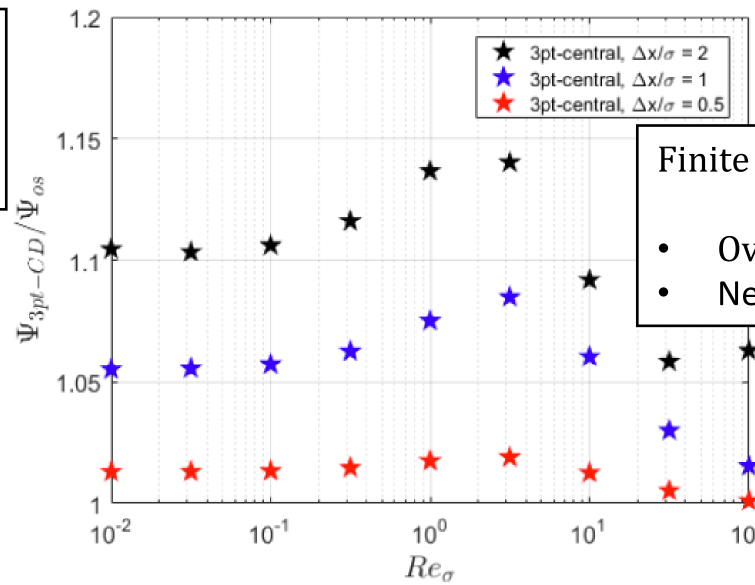
Wave number in second-order finite difference scheme

$$\tilde{k} = \frac{1}{\Delta x}\sin(k\Delta x) \text{ and } \tilde{\tilde{k}}^2 = \frac{1}{\Delta x^2}[1-\cos(k\Delta x)] \text{ for } k = 0, \pm k_{min}, \pm 2k_{min}, \cdots, \pm\frac{N}{2}k_{min}$$



Spectral

- Underestimate
- Need $\Delta x < \sigma$



Finite difference

- Overestimate
- Need $\Delta x < \sigma/2$

Conclusion

Subject: provide the self-induced perturbation velocity correction model

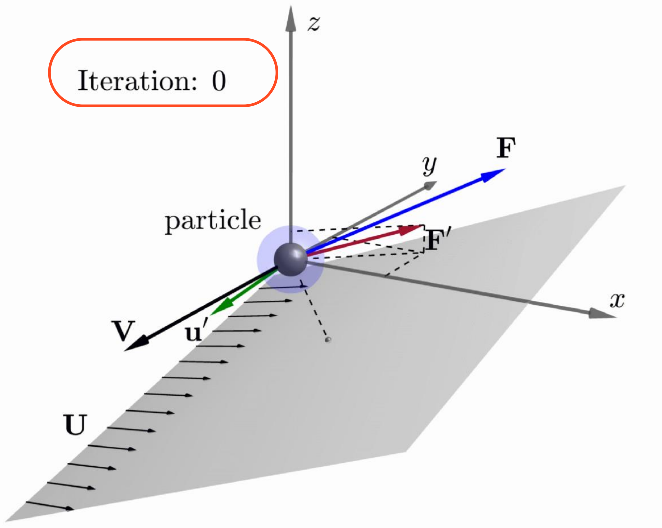
Aim: recover the true drag force in Two-way coupled Euler-Lagrange simulation
Needs: $\Delta x < \sigma$ for spectral method, $\Delta x < \sigma\,/2$ for finite difference scheme
Inputs: $\sigma, d_p, \nu, \boldsymbol{X}_w, (\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}), \left[\frac{\boldsymbol{F}'_{ot}}{\boldsymbol{F}'_{qs}}\right]$
Outputs: corrected drag

Capability:

- Arbitrary velocity vectors of flow, particle, and force
- Free-slip wall

Feature:

- Analytically derived
- Explicit form without table interpolation
- Easy to extend to different filters, DS, finite Re_σ, fast-varying, heat/mass transfer



Thank you for watching!

You can send me questions by email: kailiu@ufl.edu

Supplementary pages

Test

Inputs: $\sigma, d_p, \nu, \boldsymbol{X}_w, (\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V}), \begin{bmatrix} \boldsymbol{F}'_{ot} \\ \boldsymbol{F}'_{qs} \end{bmatrix}$

↓

Model: a black box of functions
Matlab < 100 lines, <0.01s/iteration

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Outputs: correct drag force
$\boldsymbol{F}' \rightarrow \boldsymbol{F}$

Example inputs:

$\sigma = 2$

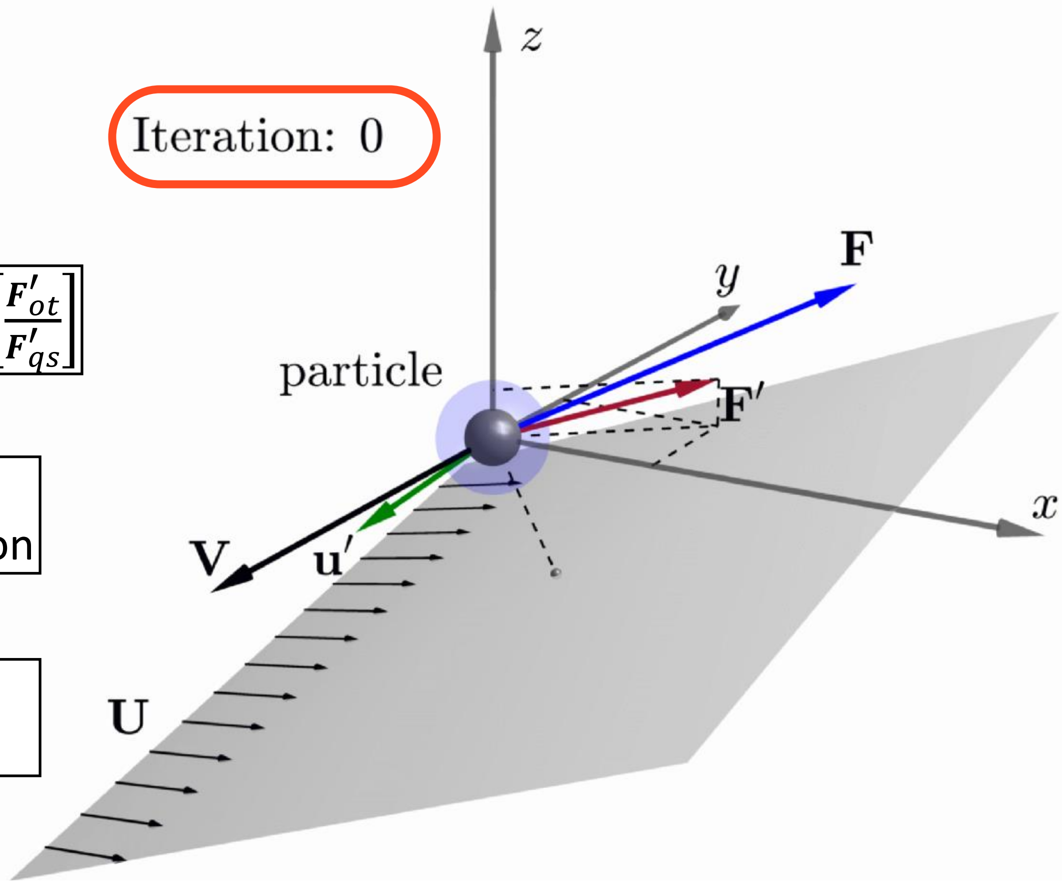
$d_p = 2$

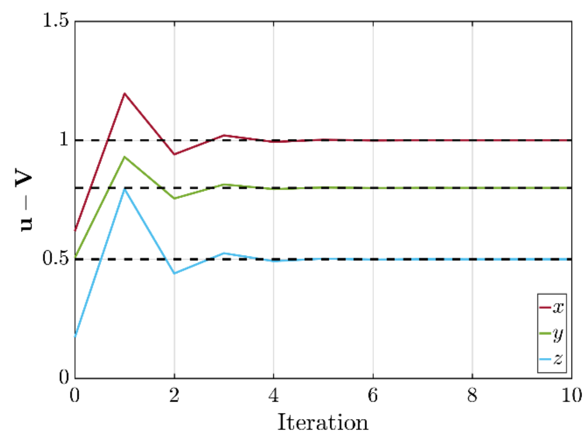
$\nu = 2$

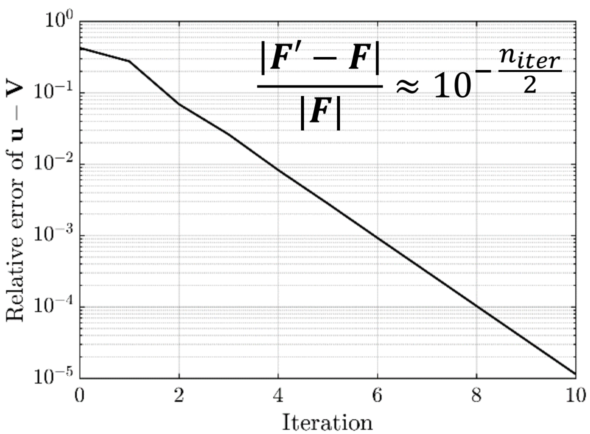
$\boldsymbol{X}_w = [2,1,-5]'$

$\overline{\phi_{c1}\boldsymbol{u}_c}^s - \boldsymbol{V} = [1,0.8,0.5]'$

$$\begin{bmatrix} \boldsymbol{F}'_{ot} \\ \boldsymbol{F}'_{qs} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$







Supplementary pages

Extension

1. Finite Reynolds number $\mathrm{Re}_\sigma \gg 1$ correction
   - Moderate derivation
   - Small self-induced velocity, less important to correct
2. Fast transient correction
   - Derive time history term
   - Commonly not used
3. Self-induced temperature/concentration corrections
   - Scalar form
   - Adiabatic/inpermeable wall